Injective types

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I will not present [1] and [2] in order.

I'll intersperse the results.

[1] M.H. Escardó. Injective types in univalent mathematics.
Mathematical structures in Computer Science
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[2] Tom de Jong & M.H.E. http://cs.bham.ac.uk/~mhe/TypeTopology
more recently, in Agda. We are unformalizing it into a paper.

$$\boxed{\text{Extension problem}}$$

embedding:

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\bar{f}"] \\ & D \end{tikzcd}$$

$j^{-1}y$ is a proposition,

$j^{-1}y \overset{\text{def}}{=} \sum_{x:X} jx = y.$ (fiber)

- D is injective if for every f and j $\underline{\text{there is some}}$ $\bar{f}$.

$\exists$

- D is algebraically injective if for any f and j $\underline{\text{there is a designated}}$ $\bar{f}$.

Σ

we denote it by

f/j as $f/j \circ j = f$.

Injectivity and excluded middle

T.F.A.E.

- excluded middle holds (i.e. we are working in a boolean topos).
- The algebraically injective types are precisely the pointed types.

Also if excluded middle holds then the injective types are precisely the non-empty types.

Injectivity under propositional resizing

(Any proposition in any universe has an equivalent copy in any universe we please.)

1. Injectivity is equivalent to the propositional truncation of algebraic injectivity. (This is a particular case of choice that just holds.)

2. The algebraically injectives are precisely the retracts of types of the form $X \to \mathcal{U}$ with $\mathcal{U}$ a type universe.

   (a) The alg. injective sets are the retracts of powersets.

   (b) The alg. inj. $(n+1)$-types are the retracts of types of the form $X \to \mathcal{U}$ with $\mathcal{U}$ a universe of n-types

ctd

3. The algebraically injective types are precisely the underlying types of the algebras of the partial-map classifier monad.

$$\mathcal{L}X \stackrel{\text{def}}{=} \sum_{P:\Omega} (P \to X)$$

type of partial elements of X.

type of propositions

4. Any universe $\mathcal{U}$ is a retract of the next universe $\mathcal{U}^+$.

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook, "s"] \arrow[dr, equal, "id"'] & \mathcal{U}^+ \arrow[d, "r"] \\ & \mathcal{U} \end{tikzcd}$$

because $\mathcal{U}$ is injective.

In the absence of propositional resizing

We are forced to keep track of universe levels.

$$
\begin{tikzcd}
\mathcal{U} & & & \mathcal{V} \\
X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & Y \arrow[dl, dotted, "f/j"] & \\
& D & & \\
& \mathcal{W} & &
\end{tikzcd}
$$

D is (algebraically) $\mathcal{U},\mathcal{V}$-injective if for every $X : \mathcal{U}$, $Y : \mathcal{V}$, $j : X \hookrightarrow Y$, $f : X \to D$, there is a (designated) f/j such that the above diagram commutes.

In the presence of propositional resizing

(Algebraic) injectivity is universe independent:

D is $\mathcal{U},\mathcal{V}$-injective iff

D is $\mathcal{U}'$-$\mathcal{V}'$-injective, for any universes $\mathcal{U},\mathcal{V},\mathcal{U}',\mathcal{V}'$.

From now on:

1. We will not assume propositional resizing.
2. We will work exclusively with algebraic injectivity, perversely abbreviated as injectivity.

(Requires univalence)

$\boxed{\text{Universes are injective}}$

In many ways. Here are two extreme ways.

Recall:

$$j^{-1}(y) \stackrel{\text{def}}{=} \sum_{x : X} fx = y$$

Left Kan extension

$$\begin{tikzcd} \mathcal{U} & X \arrow[r, hook, "j"] \arrow[d, "f"'] & Y \arrow[dl, dashed] & \mathcal{V} \\ & \mathcal{U} \sqcup \mathcal{V} & & \end{tikzcd}$$

$$f \backslash j (y) \stackrel{\text{def}}{=} \sum_{(x,-) : j^{-1} y} f x$$

proposition because j is embedding

Right Kan extension

$$\begin{tikzcd} \mathcal{U} & X \arrow[r, hook, "j"] \arrow[d, "f"'] & Y \arrow[dl, dashed] & \mathcal{V} \\ & \mathcal{U} \sqcup \mathcal{V} & & \end{tikzcd}$$

$$f / j (y) \stackrel{\text{def}}{=} \prod_{(x,-) : j^{-1} y} f x$$

$\boxed{\text{Particular case}}$ $\mathcal{U}$ is injective

$$\begin{tikzcd} \mathcal{U} & & \mathcal{U} \\ X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & Y \arrow[dl, dashed] \\ & \mathcal{U} & \end{tikzcd}$$

This doesn't work (in the absence of propositional resizing) if we promote X or Y to live in universes larger than $\mathcal{U}$.

$\Omega_{\neg\neg}$-resizing

The type $\Omega_{\neg\neg}\mathcal{U} \overset{\text{def}}{=} \sum_{X:\mathcal{U}} \text{is-prop}\, X \times (\neg\neg X \to X)$,
whose native universe is $\mathcal{U}^+$, has an equivalent copy in $\mathcal{U}$.

Not provable or disprovable.

[1] M. Rathjen. Proof theory of constructive systems: inductive types & univalence (2017)
[2] A. Swan. Double negation stable h-propositions in cubical sets arXiv (2022)

No-go theorem

There are no small injectives in general.

Suppose we have the following injectivity situation.

universe $\geq \mathcal{U}$.

$$\begin{tikzcd} \mathcal{U} \sqcup \mathcal{V} & X \arrow[r, hook, "j"] \arrow[dr, "f"'] & Y \arrow[d, dashed] & \mathcal{W} \\ & & D & \\ & & \mathcal{U} & \end{tikzcd}$$

it doesn't matter what we choose here.

has two distinct points

Then $\Omega_{\mathcal{U}}$-resizing holds.

Retracts of injectives are injective

$$
\begin{tikzcd}
\overset{u}{X} \arrow[rr, hook, "j"] \arrow[dr, "f"'] \arrow[ddr, bend right, "s\circ f"'] & & Y^{v} \arrow[dl, dashed, "f|j"] \arrow[ddl, bend left, "(s\circ f)|j"] \\
& D'_{w'} \arrow[d, shift right, "s"'] & \\
& D_{w} \arrow[u, shift right, "r"'] &
\end{tikzcd}
$$

If D is injective, then so is any retract D' of D.

Products of injectives are injective

$$\begin{tikzcd}
\mathcal{U} & & \mathcal{V} \\
X \arrow[rr, hook, "j"] \arrow[dr, "f_i"'] & & Y \arrow[dl, dashed, "f_i | j"] \\
& D_i & \mathcal{W}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{U} & & \mathcal{V} \\
X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & Y \arrow[dl, dashed, "f | j(y) = i \mapsto f_i | j(y)"] \\
& \prod_i D_i & \mathcal{W} \sqcup \mathcal{T}
\end{tikzcd}$$

Index type in universe $\mathcal{T}$.

$$\boxed{\text{Every injective type is a retract of any type it is embedded into}}$$

$$\begin{tikzcd} u & v \\ D \arrow[r, hook, "s"] \arrow[dr, "id"'] & Y \arrow[d, dashed, "r"] \\ & D \\ & u \end{tikzcd}$$

$$r \overset{\text{def}}{=} s \mid id$$

Or: every embedding of an injective type into any type is a section.

The identity type former is an embedding

Found independently by Egbert Rijke

$$X \hookrightarrow (X \to \mathcal{U})$$
$$x \longmapsto (y \longmapsto x = y)$$

Corollary. If $D : \mathcal{U}$ is $\mathcal{U}, \mathcal{U}^+$-injective then D is a retract of $D \to \mathcal{U}$.

Because

$$\begin{tikzcd} D \arrow[r, hook] \arrow[dr, "id"'] & (D \to \mathcal{U}) \arrow[d, dashed, "\pi"] \\ & D \end{tikzcd}$$

N.B. Because of the no-go theorem, the hypothesis can't be fulfilled without some form of propositional resizing.

More examples

Non-exhaustive list among the examples we know.

The following types, which live in the universe $\mathcal{U}^+$, are $\mathcal{U}$, $\mathcal{U}$-injective.

0. The type of propositions in $\mathcal{U}$.
1. The type of ordinals in $\mathcal{U}$.
2. The type of iterative (multi)sets in $\mathcal{U}$.
3. The type $\mathcal{U}_\bullet$ of pointed types in $\mathcal{U}$.
4. The types of ∞-magmas, pointed ∞-magmas, monoids, groups in $\mathcal{U}$.
5. The type $\mathcal{L}X \stackrel{\text{def}}{=} \sum p : \Omega_{\mathcal{U}}, (P \to X)$ of partial elements of any type $X : \mathcal{U}$.
6. The carriers of sup-lattices /pointed dcpos in $\mathcal{U}^+$.

Some counter-examples

(WEM) Weak excluded middle$_{\mathcal{U}}$ $\stackrel{\text{def}}{=}$ for every $P : \Omega_{\mathcal{U}}$, $\neg P$ or $\neg\neg P$.

Equivalent to De Morgan Law.

If any of the following types is injective, then WEM holds.

1. $2 \stackrel{\text{def}}{=} 1+1$.
2. The simple types, obtained from $\mathbb{N}$ by iterating "$\to$".
3. The type of Dedekind reals.
4. The type $\mathbb{N}_\infty \stackrel{\text{def}}{=} \Sigma\, \alpha : \mathbb{N} \to 2,\ (\Pi\, i : \mathbb{N}.\ \alpha_i \geq \alpha_{i+1})$.
5. Any type with an apartness relation and two points apart.
6. Any type of the form $X + Y$ with X and Y pointed.

Another counter-example

All propositions are projective$_{\mathcal{U}}$ $\overset{\text{def}}{=}$

$$\prod P : \Omega_{\mathcal{U}} \,,\ \prod Y : P \to \mathcal{U} \,,$$

$$\left(\prod p : P \,.\ \|Y_p\|\right)$$

$$\to \left\|\prod p : P \,,\ Y_p\right\| .$$

$\boxed{\text{When } \|Y_p \simeq 2\|, \text{ this is known as the world's simplest axiom of choice, which is not valid in some toposes (Fourman \& Ščedrov 1982)}}$

If the type $\sum X : \mathcal{U} \,.\ \|X\|$ of inhabited types is injective, then all propositions are projective.

An example related to the previous counterexample

The type $\Sigma\, x : \mathcal{U},\ \neg\neg X$ of non-empty types is injective.

This gives an illustration of the difference between propositional truncation and double negation.

(They are equivalent iff excluded middle holds.)

Indecomposability of injective types

In computation, it is important to identify decidable properties of types.

$\boxed{\text{Injective types have no non-trivial decidable properties.}}$

"Rice's Theorem" for injective types.

$$\text{decomposition } X \stackrel{\text{def}}{=} \Sigma f : X \to 2 ,\ f^{-1} 0 \times f^{-1} 1$$

$$\simeq \Sigma X_0, X_1 : \mathcal{U} ,\ (X_0 + X_1 \simeq X) \times X_0 \times X_1$$

If an injective type has a decomposition, then weak excluded middle holds.

When is a subtype of an injective type itself injective?

Not always. We have seen that $\mathcal{U}$ is injective, but its subtype $\Sigma X : \mathcal{U} . \| X \|$ is not in general injective.

A subtype $\Sigma d : D . P d$ of an injective type $D : \mathcal{U}^+$, with $P : D \to \Omega_{\mathcal{U}}$, is $\mathcal{U}, \mathcal{U}$-injective iff there is a designated function $f : D \to D$ s.t. for all $d : D$,

1. $P(f d)$ holds, and
2. $P d$ implies $f d = d$.

E.g.

- $f X = (\neg\neg X \to X)$ ✓
- $f X = (\| X \| \to X)$ ✗

Needs propositions projective.

(Algebraically) flabby types

A type D is (algebraically) $\mathcal{U}$-flabby if the extension problem

$$\begin{tikzcd}
\mathcal{U} & P \arrow[r, hook] \arrow[d] & 1 \arrow[dl, dashed] \\
& D & \omega
\end{tikzcd}$$

can be solved for every proposition P.

Every „partial element of D can be extended to a (total) element" of D.

Algebraic flabbinnes is witnessed by a function $\bigsqcup : (P \to D) \to D$ s.t. for every $f : P \to D$ and every $p : P$, we have $\bigsqcup f = f\, p$.

Flabby $\iff$ injective

1. The direction $\Longleftarrow$ is by definition.

   But let's record the universe levels.

   If $D : \mathcal{W}$ is $\mathcal{U}, \mathcal{V}$-injective then it is $\mathcal{U}$-flabby.

   (so notice that the universe $\mathcal{V}$ is forgotten in the conclusion.)

2. For the direction $\Longrightarrow$, we perform the following construction:

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & Y \arrow[dl, dashed] \\ & D & \end{tikzcd}$$

$$f | j\,(y) \stackrel{\text{def}}{=} \bigsqcup_{(x,-) \,:\, j^{-1}\, y} f\, x$$

is a proposition

For $D = \mathcal{U}$ we took $\bigsqcup = \Sigma$ and $\bigsqcup = \Pi$ as possible choices.

The second direction with explicit universe levels

2. If a type $D : \mathcal{W}$ is $\mathcal{U} \sqcup \mathcal{V}$-flabby then it is also $\mathcal{U}, \mathcal{V}$-injective.

We get the following resizing theorem by going back and forth with the above constructions.

$1 \Rightarrow 2 \Rightarrow 1$. If a type $D : \mathcal{W}$ is $\mathcal{U} \sqcup \mathcal{T}, \mathcal{V}$-injective, then it is also $\mathcal{U}, \mathcal{T}$-injective.

Notice that, again, the universe $\mathcal{V}$ is forgotten.

Flabbiness of Σ types

We give a sufficient condition, which we know not to be necessary (by one of the examples already given*).

① flabby

$$f : P \to X$$

$$g : \prod_{p:P} A(f\,p)$$

$$\begin{tikzcd} P \arrow[r] \arrow[dr, "{\langle f, g \rangle}"'] & 1 \arrow[d, dashed, "{(x, a)}"] \\ & \Sigma x : X .\ A x \end{tikzcd}$$

② compatibility data

The canonical map $\rho : A(\bigsqcup f) \to \prod p : P .\ A(f\,p)$ has a given section.

* The type $\mathcal{U}_\bullet$ is flabby and $\simeq \Sigma (X, -) : \underbrace{(\Sigma X : \mathcal{U} , \|X\|)}_{\text{not flabby}} , X$

Flabbiness of Σ-types of the form $\Sigma X : \mathcal{U}.\, AX$

E.g take $AX \stackrel{\text{def}}{=} (X \times X \to X)$ for the type of ∞-magmas.

Define $\tau : \Pi X, Y,\ X \simeq Y \to AX \to AY$ using univalence as expected.

Then $\tau_{X,Y}\, f$ is an equivalence $AX \simeq AY$.

Lemma. For any T of the same type as τ,

if $T\, \mathrm{id} \sim \mathrm{id}$, then $T_f \sim \tau_f$ for any $f : X \simeq Y$.

$\boxed{\text{ctd.}}$

For any $p_0 : P$ we have a canonical equivalence $\left(\prod p : P.\ F p\right) \overset{\varphi}{\simeq} F p_0$.

$P : \Omega$ and $F : P \to \mathcal{U}$

Lemma. The compatibility condition holds if the following map has a section for all $P : \Omega_{\mathcal{U}}$ and $F : P \to \mathcal{U}$:

$$\begin{aligned} p : A\left(\textstyle\prod_{p : P}.\ F p\right) &\longrightarrow \textstyle\prod p : P.\ A\ F p \\ a &\longmapsto (p_0 \mapsto T \varphi\, a) \end{aligned}$$

This condition is very easy to check for magmas, monoids groups etc

The only "art" is to choose a suitable T with $T\ \mathrm{id} \sim \mathrm{id}$.

Discussion

1. My original interest in injectivity comes from my work on searchable types in which I use injectivity to construct plently of them, using ordinals to measure their complexity. These types are totally separated (the functions into 2 separate the points) even though injective types are indecomposable.
2. There are equations for algebraic flabbiness that make the notion even more algebraic, in fact coinciding with algebra of the partial-map classifier monad $\mathcal{L}$.
3. while a paper is not avilabe, we invite you to read TypeTopology. :)

$$\begin{tikzcd}
\mathbb{N} \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{N}_\infty \arrow[d, dashed, "{\Sigma, \Pi}"] \\
& M
\end{tikzcd}$$

want this

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[dr, "f"'] & Y \arrow[r, hook, "k"] \arrow[d, "f|j"] & Z \arrow[dl] \\
& D &
\end{tikzcd}$$

$$(f|j)|k = f|(k \circ j)$$

$$\begin{tikzcd}
P \arrow[r] \arrow[dr] & 1 \arrow[d] \\
& D
\end{tikzcd}$$

$$\bigsqcup$$

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[dr, "f"'] & Y \arrow[d, dashed] \\
& D
\end{tikzcd}$$

$$f|j \overset{?}{=} y \mapsto \bigsqcup_{(x,-) : j^{-1} y} f\, x$$

(written during question time)