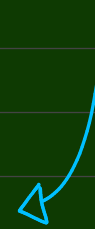


Injective types

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I will not present
[1] and [2] in order.

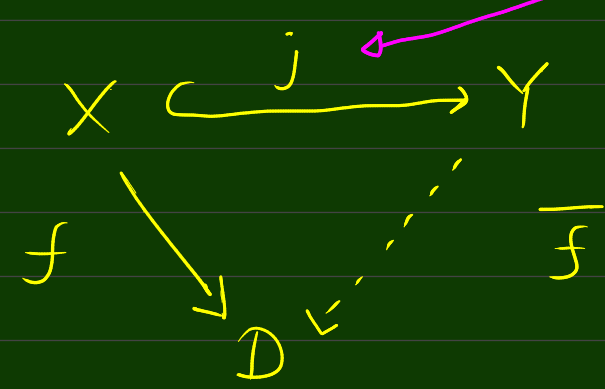
I'll intersperse the
results.



[1] M.-H. Escardó. Injective types in univalent mathematics.
Mathematical structures in Computer Science
Vol 31, Issue 1, pp. 89-111, 2021

[2] Tom de Jong & M.-H.E. <http://cs.bham.ac.uk/~mhe/TypeTopology>
more recently, in Agda. We are unformalizing it into a paper.

Extension problem



embedding:

$j^{-1}y$ is a proposition.

$$j^{-1}y \stackrel{\text{def}}{=} \sum_{x:X} jx = y. \quad (\text{fiber})$$

\exists

- D is injective if for every f and j there is some \bar{f} .
- D is algebraically injective if for any f and j there is a designated \bar{f} .

Σ

we denote it by f/j as $f/j \circ j = f$.

Injectivity and excluded middle

T.F.A.E.

- excluded middle holds (i.e. we are working in a boolean topos).
- The algebraically injective types are precisely the pointed types.

Also if excluded middle holds then the injective types are precisely the non-empty types.

Injectivity under propositional resizing

(Any proposition in any universe has an equivalent copy in any universe we please.)

1. Injectivity is equivalent to the propositional truncation of algebraic injectivity. (This is a particular case of choice that just holds.)
2. The algebraically injectives are precisely the retracts of types of the form $X \rightarrow \mathcal{U}$ with \mathcal{U} a type universe.
 - (a) The alg. injective sets are the retracts of powersets.
 - (b) The alg. inj. $(n+1)$ -types are the retracts of types of the form $X \rightarrow \mathcal{U}$ with \mathcal{U} a universe of n -types

ctd

3. The algebraically inductive types are precisely the underlying types of the algebras of the partial-map classifier monad.

$$\mathcal{L}X \stackrel{\text{def}}{=} \sum_{P:\Omega} (P \rightarrow X)$$

← type of partial elements of X .

↑ type of propositions

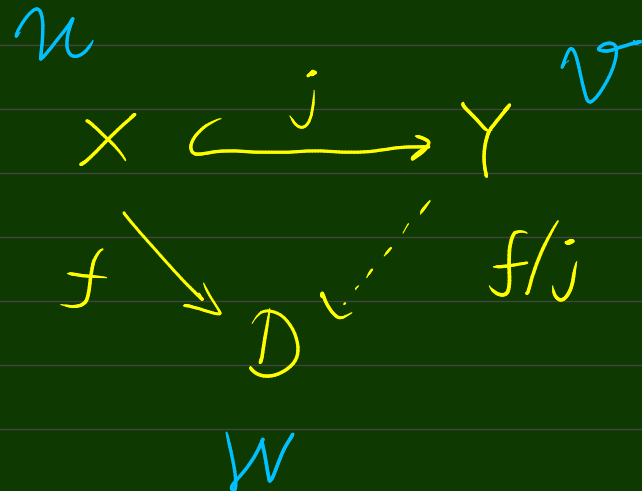
4. Any universe \mathcal{U} is a retract of the next universe \mathcal{U}^+ .

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\iota} & \mathcal{U}^+ \\ \text{id} \searrow & & \swarrow \pi \\ & \mathcal{U} & \end{array}$$

because \mathcal{U} is inductive.

In the absence of propositional resizing

We are forced to keep track of universe levels.



D is (algebraically) u, v -injective if for every $x:u, y:v$,

$j: X \hookrightarrow Y, f: X \rightarrow D$, there is a (designated) f/j

such that the above diagram commutes.

In the presence of propositional resizing

(Algebraic) injectivity is universe independent:

D is u, v -injective iff

D is u', v' -injective, for any universes u, v, u', v' .

From now on:

1. We will **not** assume propositional resizing.
2. We will work exclusively with algebraic injectivity, perversely abbreviated as injectivity.

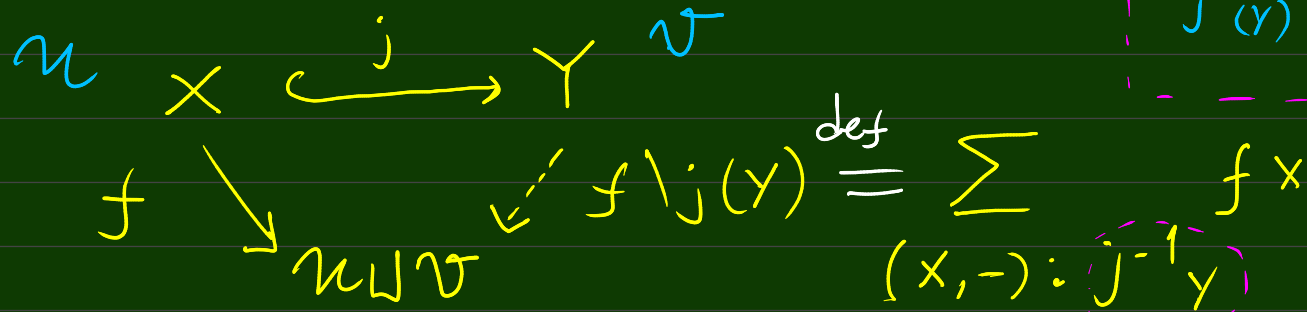
(Requires univalence)

Universes are inductive

In many ways. Here are two extreme ways.

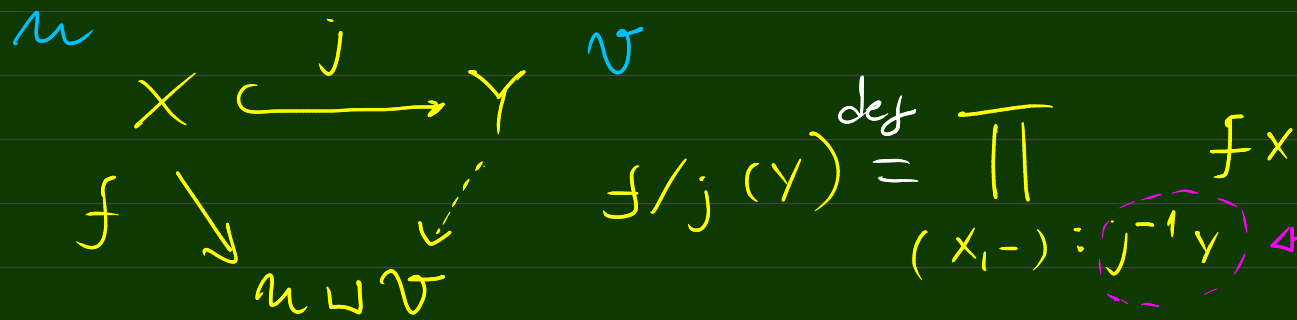
Recall:
 $j^{-1} \stackrel{\text{def}}{=} \sum_{x: X} f x = y$

Left Kan extension



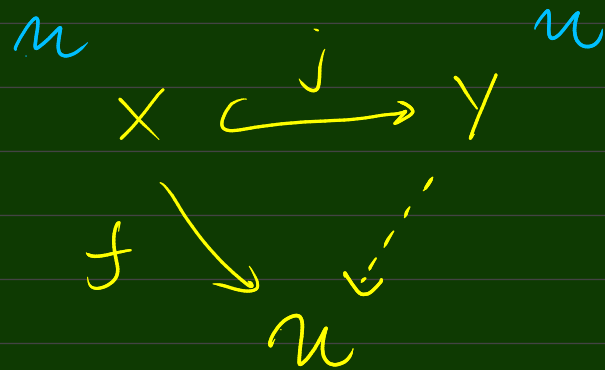
proposition because j is embedding

Right Kan extension



Particular case

\mathcal{U} is injective



This doesn't work (in the absence of propositional resizing)

if we promote X or Y to live in universes larger than \mathcal{U} .

$\Omega_{\neg\neg}$ -resizing

The type $\Omega_{\neg\neg} \mathcal{U} \stackrel{\text{def}}{=} \sum_{x:\mathcal{U}} \text{is-prop } X \times (\neg\neg X \rightarrow X)$,
whose native universe is \mathcal{U}^+ , has an equivalent copy in \mathcal{U} .

Not provable or disprovable.

[1] M. Rathjen. Proof theory of constructive systems: inductive types & univalence (2017)

[2] A. Swan. Double negation stable h-propositions in cubical sets arXiv (2022)

No-go theorem

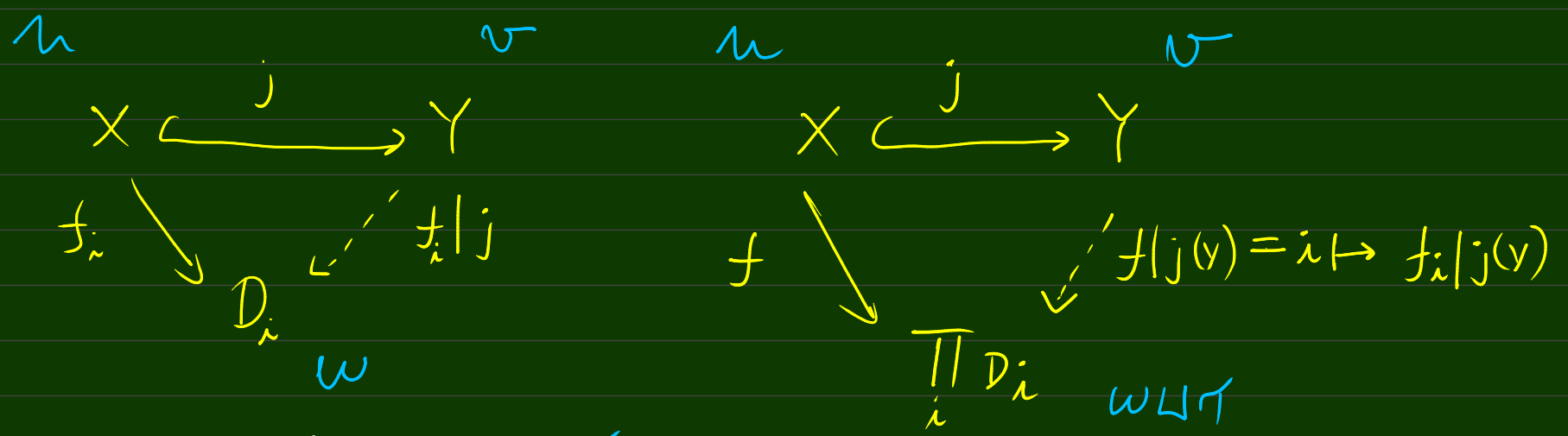
There are no small inductives in general.

Suppose we have the following inductivity situation.



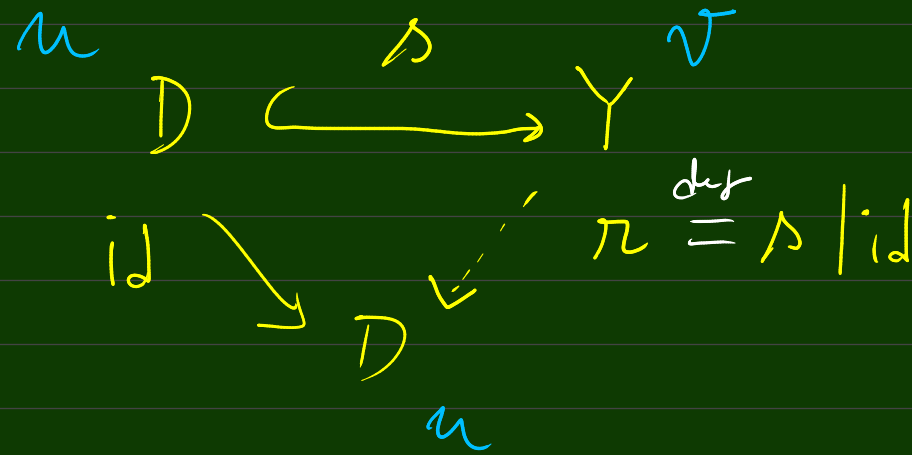
Then $\Omega_{\mathcal{U}}$ -resizing holds.

Products of injectives are injective



Index type in universe \mathcal{T} .

Every injective type is a retract of any type it is embedded into



or: every embedding of an injective type into any type is a section.

The identity type former is an embedding

Found independently by Egbert Rijke

$$X \hookrightarrow (X \rightarrow \mathcal{U})$$

$$x \longmapsto (y \longmapsto x = y)$$

Corollary. If $D: \mathcal{U}$ is $\mathcal{U}, \mathcal{U}^+$ -injective then D is a retract of $D \rightarrow \mathcal{U}$.

Because

$$\begin{array}{ccc} D & \hookrightarrow & (D \rightarrow \mathcal{U}) \\ \text{id} \searrow & & \swarrow \tau \\ & D & \end{array}$$

N.B. Because of the no-go theorem, the hypothesis can't be fulfilled without some form of propositional resizing.

More examples

Non-exhaustive list among the examples we know.

The following types, which live in the universe \mathcal{U}^+ , are \mathcal{U}, \mathcal{U} -injective.

0. The type of propositions in \mathcal{U} .
1. The type of ordinals in \mathcal{U} .
2. The type of iterative (multi)sets in \mathcal{U} .
3. The type \mathcal{U}_\bullet of pointed types in \mathcal{U} .
4. The types of ∞ -magmas, pointed ∞ -magmas, monoids, groups in \mathcal{U} .
5. The type $\mathcal{P}X \stackrel{\text{def}}{=} \sum_{P: \Omega_{\mathcal{U}}} (P \rightarrow X)$ of partial elements of any type $X: \mathcal{U}$.
6. The carriers of s-plattices in \mathcal{U}^+ .
/pointed dcpos

Some counter-examples

(WEM) Weak excluded middle $\stackrel{\text{def}}{=} \text{for every } P: \Omega_u, \neg P \text{ or } \neg\neg P.$

Equivalent to De Morgan Law.

If any of the following types is inductive, then WEM holds.

1. $2 \stackrel{\text{def}}{=} 1+1.$

2. The simple types, obtained from \mathbb{N} by iterating " \rightarrow ".

3. The type of Dedekind reals.

4. The type $\mathbb{N}_\infty \stackrel{\text{def}}{=} \sum \alpha: \mathbb{N} \rightarrow 2, (\prod i: \mathbb{N}, \alpha_i \geq \alpha_{i+1}).$

5. Any type with an apartness relation and two points apart.

6. Any type of the form $X + Y$ with X and Y pointed.

Another counter-example

All propositions are projective def

$$\prod P = \Omega_U, \prod Y: P \rightarrow U,$$

$$(\prod P = P, \|Y_P\|)$$

$$\rightarrow \| \prod P = P, Y_P \|.$$

When $\|Y_P \simeq \Omega\|$, this is known as the world's simplest axiom of choice, which is not valid in some toposes (Fourman & Šcedrov 1982)

If the type $\sum x:U. \|x\|$ of inhabited types is inductive, then all propositions are projective.

An example related to the previous counter example

The type $\sum x:\mathbb{N}, \neg\neg x$ of non-empty types is inductive.

This gives an illustration of the difference between propositional truncation and double negation.

(They are equivalent iff excluded middle holds.)

Indecomposability of inductive types

In computation, it is important to identify decidable properties of types.

Inductive types have no non-trivial decidable properties.

"Rice's Theorem"
for inductive
types.

decomposition $X \stackrel{\text{def}}{=} \sum f: X \rightarrow \mathbb{Z}, f^{-1}0 \times f^{-1}1$

$\simeq \sum X_0, X_1: \mathcal{U}, (X_0 + X_1 \simeq X) \times X_0 \times X_1$

If an inductive type has a decomposition,
then weak excluded middle holds.

When is a subtype of an inductive type itself inductive?

Not always. We have seen that \mathcal{N} is inductive, but its subtype $\sum x:\mathcal{N} . \|x\|$ is not in general inductive.

A subtype $\sum d:D . P d$ of an inductive type $D:\mathcal{U}^+$, with $P:D \rightarrow \Omega_{\mathcal{U}}$, is \mathcal{U},\mathcal{U} -inductive iff

there is a designated function $f:D \rightarrow D$ s.t. for all $d:D$,

1. $P(f d)$ holds, and
2. $P d$ implies $f d = d$.

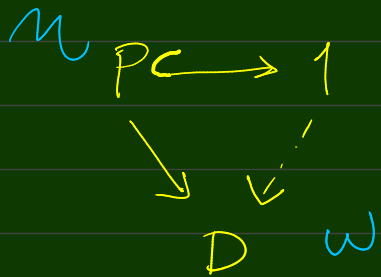
E.g.

- $f x = (\neg x \rightarrow x)$ ✓
- $f x = (\|x\| \rightarrow x)$ ✗

Needs propositions projective.

(Algebraically) flabby types

A type D is (algebraically) \mathcal{U} -flabby if the extension problem



can be solved for every proposition P .

Every partial element of D can be extended to a (total) element of D .

Algebraic flabbiness is witnessed by a function $\sqcup : (P \rightarrow D) \rightarrow D$ s.t. for every $f : P \rightarrow D$ and every $p : P$, we have $\sqcup f = f p$.

Flabby \iff injective

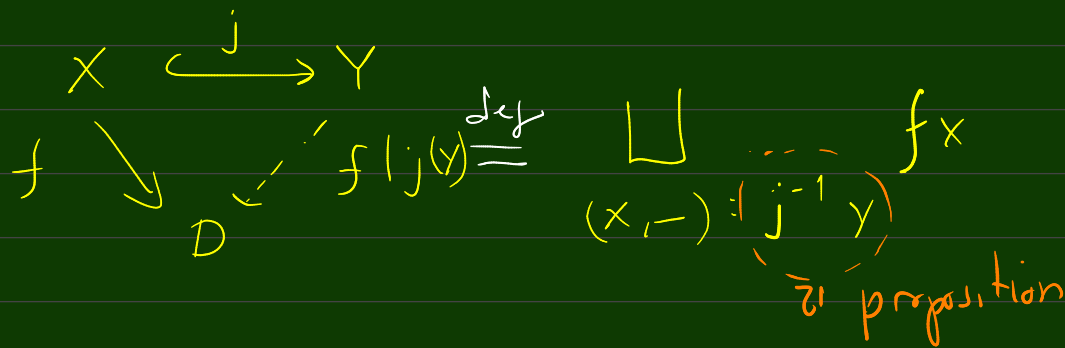
1. The direction \Leftarrow is by definition.

But let's record the universe levels.

If $D : W$ is u, v -injective then it is u -flabby.

(so notice that the universe v is forgotten in the conclusion.)

2. For the direction \Rightarrow , we perform the following construction.



For $D = \mathcal{U}$ we took $W = \Sigma$ and $\sqcup = \prod$ as possible choices.

The second direction with explicit universe levels

2. If a type $D:W$ is $u \sqcup v$ -flabby then it is also u, v -injective.

We get the following resizing theorem by going back and forth with the above constructions.

$1 \Rightarrow 2 \Rightarrow 1$. If a type $D:W$ is $u \sqcup \tau, v$ -injective, then it is also u, τ -injective.

Notice that, again, the universe V is forgotten.

Flabbiness of Σ types

We give a sufficient condition, which we know not to be necessary (by one of the examples already given^{*}).



$$f: P \rightarrow X$$

$$g: \prod_{p:P} A(f p)$$

① flabby

② compatibility data

The canonical map $\rho: A(\coprod f) \rightarrow \prod_{p:P} A(f p)$ has a given section.

* The type \mathcal{U}_0 is flabby and $\simeq \Sigma(x, -) : \left(\underbrace{\Sigma_{x:\mathcal{U}_0} \mathbb{N}}_{\text{not flabby}}, |X| \right), X$

Flabbiness of Σ -types of the form $\Sigma x:U. AX$

E.g take $AX \stackrel{\text{def}}{=} (X \times X \rightarrow X)$ for the type of ∞ -magnets.

Define $\tau: \prod X, Y, X \simeq Y \rightarrow AX \rightarrow AY$ using univalence as expected.

Then $\tau_{x,y} \circ f$ is an equivalence $AX \simeq AY$.

Lemma. For any T of the same type as τ ,

if $T \text{ id} \sim \text{id}$, then $T f \sim \tau f$ for any $f: X \simeq Y$.

ctd.

$P: \Omega$
For any $P_0: P$ we have a canonical equivalence $(\overline{\Pi}_{P:P} \cdot F_P) \stackrel{\varphi}{\cong} F_{P_0}$.
and $F: P \rightarrow \mathcal{U}$

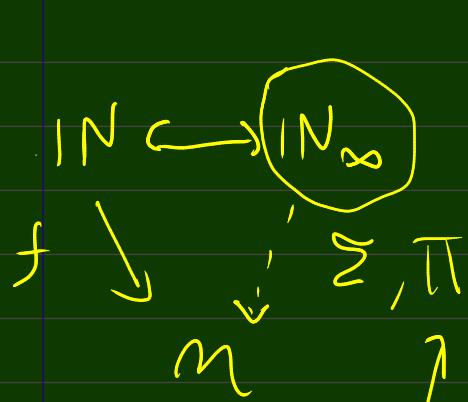
Lemma. The compatibility condition holds if the following map has a section for all $P: \Omega \mathcal{U}$ and $F: P \rightarrow \mathcal{U}$:

$$P: A(\overline{\Pi}_{P:P} \cdot F_P) \rightarrow \overline{\Pi}_{P:P} \cdot A F_P \\ a \longmapsto (P_0 \mapsto T \varphi a)$$

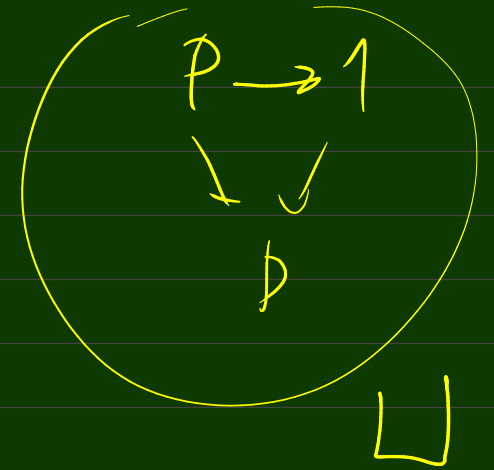
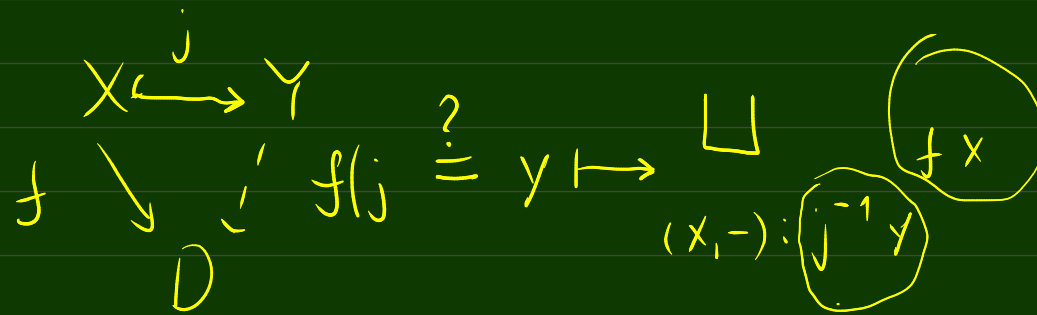
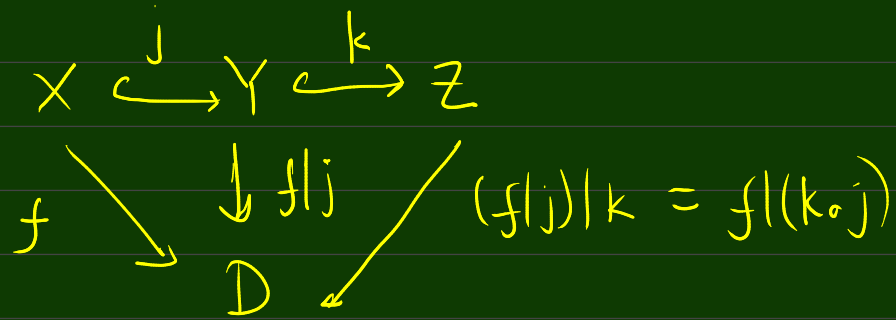
This condition is very easy to check for monoids, monoids groups etc.
The only "art" is to choose a suitable T with $T \text{id} \sim \text{id}$.

Discussion

1. My original interest in injectivity comes from my work on **searchable types** in which I use injectivity to construct plenty of them, using ordinals to measure their complexity.
These types are **totally separated** (the functions into 2 separate the points) even though injective types are **indecomposable**.
2. There are **equations for algebraic flatness** that make the notion even more algebraic, in fact coinciding with algebra of the partial-map classifier monad \mathcal{L} .
3. While a paper is not available, we invite you to read **TypeTopology**. 😊



want this



(written during question time)