The geometry of constancy

(in HoTT and in `cubicaltt`)

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Exiting propositional truncations

Often we have $\|X\| \to X$, even when we don't know whether X is empty or inhabited.

E.g. For any $f : \mathbb{N} \to \mathbb{N}$, we have $\| \sum_{n:\mathbb{N}} fn = 0\| \to \sum_{n:\mathbb{N}} fn = 0$.

However, global choice

$$\prod_{X:U} \|X\| \to X$$

implies that all types have decidable equality and hence negates univalence.

Theorem (with Nicolai, Thierry and Thorsten):

> *There is a choice function $\|X\| \to X$ iff there is a constant endo-map $X \to X$.*

Question:

> *Can we eliminate $\|X\| \to A$ using a constant map $X \to A$?*

Two answers: Yes (Nicolai Kraus) and no (Mike Shulman).

Nicolai considers coherently constant functions.
Mike considers arbitrary constant functions.

Constancy

1. A function $f : X \to A$ is constant if any two of its values are equal.

$$\text{constant}\, f \stackrel{\text{def}}{=} \prod_{x,y:X} fx = fy.$$

2. This is "structure" or data rather than property, unless A is a set.

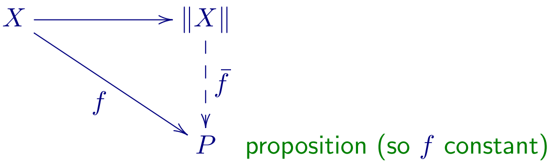
   A function can be constant in zero, one or more ways.

3. E.g. the function $f : 1 \to S^1$ with definitional value base has $\mathbb{Z}$-many moduli of constancy $\kappa_n : \text{constant}\, f$:

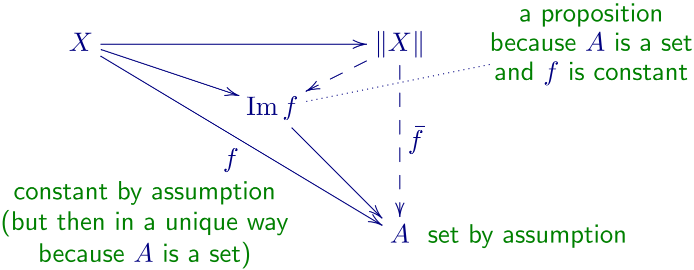
$$\kappa_n(x)(y) \stackrel{\text{def}}{=} \text{loop}^n\,.$$

Set-valued constant functions

1. For any proposition P, by definition of truncation:

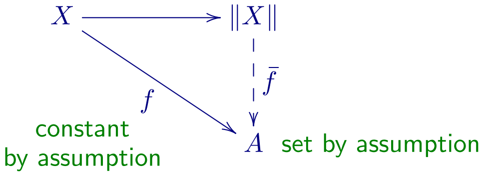


2. Can replace P by a set A:



Propositional truncation as a set quotient

1. I.e. $\|X\|$ is the set-quotient of X by the chaotic relation:



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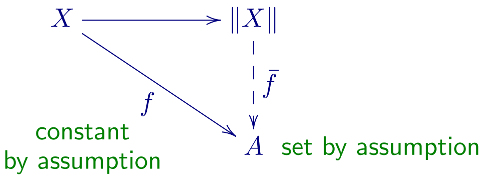
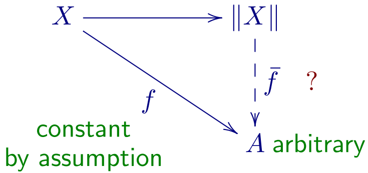
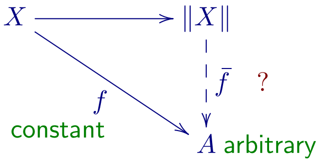


2. Can we replace A by an arbitrary type?



No, not in general (Shulman, `http://homotopytypetheory.org/2015/06/11/not-every-weakly-constant-function-is-conditionally-constant/`)

When do we get a factorization of a constant function?



The factorization is possible if any of the following conditions holds:

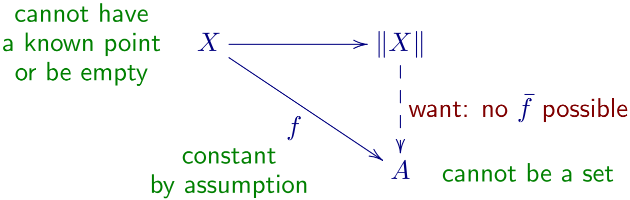
1. X is empty.
2. X has a given point.
3. We have a function $\|X\| \to X$.
4. We have a function $A \to X$.
5. A is a set.

What other sufficient conditions?

And what about necessary conditions?

Also, given any factorization, we can construct another one for which the triangle commutes judgementally.

How to construct a counter example



Natural attempt to get a counter-example

Let $s : S^1$ be an arbitrary point of the circle.

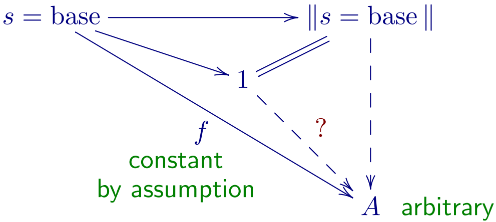
Let A be an arbitrary type.

Let $f : s = \mathrm{base} \to A$ be constant.

We can't know a point of the path space $s = \mathrm{base}$ in general.

But we know it is inhabited, that is, $\| s = \mathrm{base} \|$

Hence $\| s = \mathrm{base} \| = 1$ by propositional univalence/extensionality.



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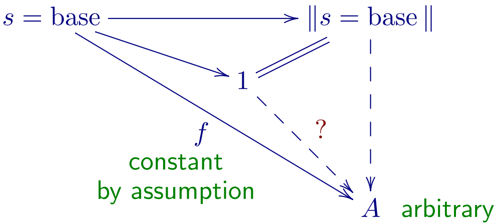

Can we expect to be able to get a point of an arbitrary type A,

from any given constant function $f : s = \text{base} \to A$,

even though we can't expect to get a point of $s = \text{base}$ in general?

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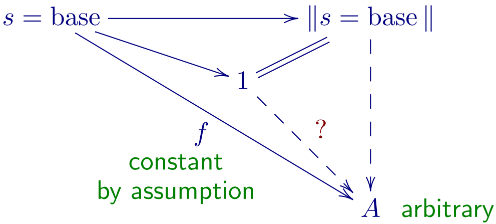

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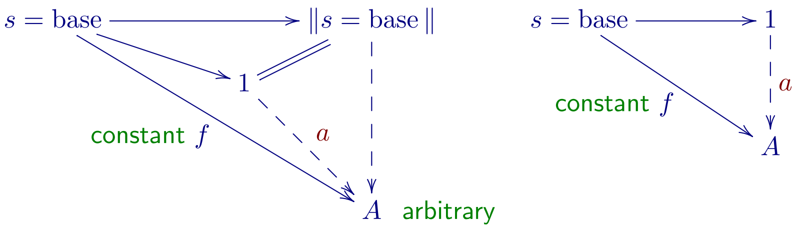
To our surprise, we can.

The attempt fails.

Theorem/Construction



For any $s : S^1$ and any constant function $f : s = \mathrm{base} \to A$ into an arbitrary type, we can find $a : A$ such that $fp = a$ for all $p : s = \mathrm{base}$.

$$\prod_{s:S^1} \prod_{A:U} \prod_{f:s=\mathrm{base}\to A} \mathrm{constant}\, f \to \sum_{a:A} \prod_{p:s=\mathrm{base}} fp = a.$$

Proof outline

1. First show that for any given family of constant functions

$$f : \prod_{s:S^1} s = \mathrm{base} \to A(s),$$

each of them factors through 1. We get $\bar{f} : \prod_{s:S^1} A(s)$

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Proof outline

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   each of them factors through $\mathbf{1}$. We get $\bar{f} : \prod_{s:S^1} A(s)$

   This allows us to use induction on the circle and on paths.

2. For any type X, consider the universal constant map on X, $\beta_X : X \to S(X)$, constructed as a HIT.

3. By (1) applied to the family $\beta_s : s = \mathrm{base} \to S(s = \mathrm{base})$ given by (2), we get a function $\bar{\beta} : \prod_{s:S^1} S(s = \mathrm{base})$.

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4. Now, given a single constant function $f : s = \mathrm{base} \to A$, it factors through the universal constant map $\beta_s : s = \mathrm{base} \to S(s = \mathrm{base})$ as $f' : S(s = \mathrm{base}) \to A$ by (2), and hence we get the required point of A as using (3), as $f'(\bar{\beta}(s))$.

Step 1

For any $f : \prod_{s:S^1} s = \mathrm{base} \to A(s)$, with $f\ \mathrm{base}$ constant, there is $\bar{f} : \prod_{s:S^1} A(s)$ such that $f\, s\, p = \bar{f}\, s$ for all $p : s = \mathrm{base}$.

1. Lemma Any transport of a value of f is a value of f:

$$\prod_{b,b':S^1} \prod_{r:b=b} \prod_{l:b=b'} \sum_{q:b'=b} \mathrm{transport}\, l\, (f\, b\, r) = f\, b'\, q.$$

   This doesn't depend on the fact that S^1 is the circle or on the constancy of $f\ \mathrm{base}$, and has a direct proof by based path induction.

2. We are interested in this particular case:

$$\sum_{q:\mathrm{base}=\mathrm{base}} \mathrm{transport\,loop}\, (f\ \mathrm{base}\, (\mathrm{refl\,base})) = f\ \mathrm{base}\ q.$$

3. Then the constancy of $f\ \mathrm{base}$ gives

$$\mathrm{transport\,loop}\, (f\ \mathrm{base}\, (\mathrm{refl\,base})) = f\ \mathrm{base}\, (\mathrm{refl\,base}),$$

   which makes S^1-induction work.

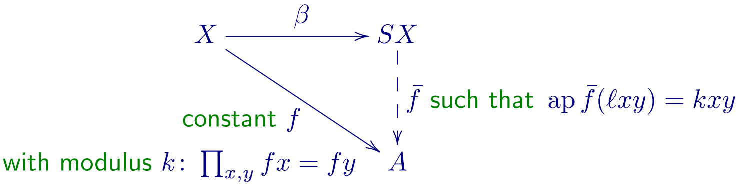
Step 2

For any type X, consider the universal constant map on X,

$$\beta : X \to S(X),$$

defined as a HIT with higher constructor

$$\ell : \prod_{x,y:X} \beta x = \beta y.$$



When X is the terminal type 1, we get the circle S^1.

Universal property of the constancy HIT

$$\begin{aligned} \beta &: X \to S(X), \\ \ell &: \prod_{x,y:X} \beta x = \beta y. \end{aligned}$$

There is an equivalence

$$\begin{aligned} SX \to A &\cong \sum_{f:X\to A} \text{constant } f \\ g &\mapsto (g \circ \beta,\ \lambda xy.\, \mathrm{ap}\, g\, (\ell xy)). \end{aligned}$$

This generalizes the universal property of the circle

$$\begin{aligned} S^1 \to A &\cong \sum_{a:A} a = a \\ &\cong \sum_{f:1\to A} \text{constant } f. \end{aligned}$$

Side remark

(Not used in the proof, at least not explicitly.)

1. The universal constant map $\beta_X : X \to S(X)$ is a surjection.
2. The type $S(X)$ is conditionally connected, meaning that

$$\prod_{s,t:S(X)} \|s = t\|.$$

   ("Conditionally" because it is empty if (and only if) X is empty.)

`cubicaltt` proof

Demonstrate and discuss some fragments of the `geometryOfConstancy.ctt` file (on my papers web page).

The constant factorization problem

Because the universal map $X \to \|X\|$ into a proposition is constant (in a unique way), the universal property of $S(X)$ gives a function

$$\prod_{X:U} S(X) \to \|X\|.$$

The existence of a function in the other direction,

$$\prod_{X:U} \|X\| \to S(X),$$

is equivalent to the statement that all constant functions $f : X \to A$ factor through $X \to \|X\|$.

But we know that this is not the case, by Shulman's construction.

However, this does hold for $X = (s = \mathrm{base})$ and all A.

Step 3

By (1) applied to the family $\beta_s : s = \mathrm{base} \to S(s = \mathrm{base})$ of constant functions given by (2), we get a function

$$\bar{\beta} : \prod_{s:S^1} S(s = \mathrm{base}).$$

This is perhaps surprising, because we don't have, of course,

$$\prod_{s:S^1} s = \mathrm{base},$$

as that would mean that that the circle is contractible.

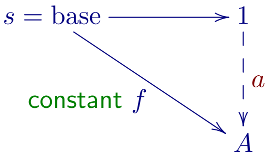
How come we are able to pick a point of the generalized circle $S(s = \mathrm{base})$, without being able to pick a point of the path space $s = \mathrm{base}$, naturally in $s : S^1$?

Step 4

Now, given a single constant function $f : s = \mathrm{base} \to A$, it factors through the universal constant map $\beta_s : s = \mathrm{base} \to S(s = \mathrm{base})$ as $f' : S(s = \mathrm{base}) \to A$ by (2), and hence we get the required point of A using (3), as

$$a \overset{\mathrm{def}}{=} f'(\bar{\beta}(s)).$$

Theorem



Conjecture

In a type theory with $\| - \|$ and (hence) function extensionality.

All constant functions $f : X \to A$ *of any two types factor through* $X \to \|X\|$ *if and only if all types are sets (zero-truncated).*

And hence univalence fails if all constant functions factor through the truncation of their domains.

(Shulman's construction exhibits a family of constant functions such that if all of them factor through the truncation of their domain, then univalence fails.)