The intrinsic topology of a universe in intuitionistic type theory

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Martin-Löf's intuitionistic type theory

1. A programming language with fancy types.

2. An alternative to ZFC, for constructive mathematics.

   E.g. it is claimed that it can formalize Bishop mathematics.

3. Variations/extensions implemented as NuPrl, Coq, Lego, Agda, . . .

4. Like set theory, ML type theory can be treated *naively* rather than *formally*.

   With the understanding that whatever we do should be formalizable.

   This is the approach taken in this talk. (But I have formalized the claims.)

The universe U is a large type of small types

Some typical uses of the universe include:

1. U is the collection of propositions. E.g. excluded middle is expressed as

$$\prod_{A : U} A + (A \to \mathbf{0}).$$

2. Large type of small types with a distinguished element and a binary operation:

$$\sum_{X:U} X \times (X \times X \to X).$$

3. Large type of small monoids:

$$\mathrm{Monoid} = \sum_{X:U} \sum_{e:X} \sum_{(\cdot):X \times X \to X} [e \text{ and } \cdot \text{ have suitable properties}].$$

A question by Vladimir Voevodsky (17 Oct 2011)

1. The topological topos is a category of continuous maps of space-like objects.

   Constructed by Johnstone in 1979.

   Related to a model by Normann and Waagbø 2002.

2. Being a topos, it is a model of Martin-Löf type theory.

3. A construction by Thomas Streicher (2004) gives an interpretation of the universe.

Question. What is this space that interprets the universe?

A possible reading of this question is:

Question. What is the topology of Streicher's universe in the topological topos?

I instead answer a different question in this talk

1. Work *within* Martin-Löf type theory.

   Like classical mathematicians work within ZFC.

2. Formulate a (model-independent) topological question for any universe.

3. Define intrinsic sequence convergence in the language of Martin-Löf type theory.

Question. What are the intrinsically convergent sequences of types in the universe?

The answer will be perhaps shocking

Theorem of Martin-Löf type theory. Every sequence of types converges to any type.

This means that the universe of types is intrinsically indiscrete.

There can only be non-trivial functions into other *indiscrete spaces*.

$U \to U$ and $U \times U \to U$ and $U \times U \to U \times U$ have plenty.

$U \to \mathbb{N}$ and $U \to 2$ have only trivial ones, because $\mathbb{N}$ and 2 are *discrete*.

But let's not neglect Voevodsky's original question

1. The *Sierpinski space* $\mathbb{S}$ is the open-subspace classifier.
2. It has an isolated point $\top$ and a limit point $\bot$.
   - The sequences that converge to $\top$ are the eventually constant ones.
   - Every sequence converges to $\bot$.
3. The Sierpinski space lives in the topological topos.
4. Let U be Streicher's universe in the topological topos.

Encouraged by the shocking answer, I posed the following to Thomas Streicher:

Conjecture. All maps $U \to \mathbb{S}$ in the topological topos are constant.

Th universe has only two open sets.

This would mean that the topological reflection of U is indiscrete.

Theorem (Streicher). The conjecture is indeed true

Streicher's universe in the topological topos has only two open subspaces.

Streicher's universe is topologically indiscrete.

Martin-Löf theory

1. A type of natural numbers $\mathbb{N}$ that supports primitive recursion.
2. Types $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ with zero, one and two elements.
3. Products $\prod$, sums Σ, identity types $=$.
4. In particular, products $\times$, sums $+$, and function types $\to$.
5. A universe U closed under the above constructions, regarded as a large type.

And perhaps more constructions and axioms (e.g. W-types, univalence, . . .).

BHK interpretation of logic

A proposition is identified with its type of "witnesses" or "realizers" or "proofs":

1. $A \wedge B = A \times B$,
2. $A \vee B = A + B$,
3. $A \Rightarrow B = A \to B$,
4. $\neg A = A \to \mathbf{0}$,
4. $\forall x \colon X (A(x)) = \prod_{x \colon X} A(x)$,
5. $\exists x \colon X (A(x)) = \sum_{x \colon X} A(x)$.

Rather than the truth of a proposition, one considers type inhabitation.

Propositional equality, or identity type

1. For any type X and elements $x, y : X$, we have the identity type $\boxed{x =_X y}$.

2. Intuitively, the type $x = y$ has at most one element.

   It is a singleton if and only if x and y are the same.

   As usual, the intuition is wrong and too naive, but still often successful.

3. More precisely, $=_X$ is the least reflexive relation.

   This is formulated by a certain induction principle J.

   This level of precision is not needed for this talk. The intuition will do.

   But be warned.

Axiom of extensionality

1. We have $f = g \implies \forall x \colon X (f(x) = g(x))$.

   Theorem of Martin-Löf type theory.

2. Don't have $\forall x \colon X (f(x) = g(x)) \implies f = g$ or its negation.

   Axiom of extensionality.

Extensionality is undecided in Martin-Löf type theory.

Voevodsky's univalence decides it positively. But univalence itself is undecided.

Voevodky's univalence axiom

A strengthening of the statement that any two isomorphic types are equal.

- Fails in the set-theoretical model of types.
- Holds in the model of simplicial sets.

This axiom underlies the view that Martin-Löf types are homotopy types.

Homotopy type theory.

Amazingly, univalence implies the axiom of extensionality.

Excluded middle

The proposition $\forall A\colon U(A \vee \neg A)$ amounts to $\prod_{A\colon U}(A + (A \to \mathbf{0}))$.

1. It cannot be proved. (Because cannot be realized in a recursive model.)
2. Its negation cannot be proved either. (Because classical sets are a model.)

We are happy to keep it undecided (hence compatible with both models).

Like one does in Bishop mathematics.

WLPO, a relevant special case of excluded middle

The weak limited principle of omniscience.

Every binary sequence is constantly one or it isn't.

$$\forall \alpha \colon 2^{\mathbb{N}} \; (\forall n \colon \mathbb{N}(\alpha_n = 1) \;\vee\; \neg \forall n \colon \mathbb{N}(\alpha_n = 1)).$$

Also independent for the same reasons.

(In the recursive model it solves the Halting Problem.)

We are happy to also keep it undecided.

But **it is a constructive taboo.**

This is the end of the introduction and background

Dream

A sequence x_n converges to a limit l if and only if $x_\infty = l$.

When you wake up in the morning, you realize that it doesn't make sense to write x_∞.

After you have a cup of coffee, you fix this as follows

Extend the type $\mathbb{N}$ to a type $\mathbb{N}_\infty$ so that:

1. The sequence $0, 1, 2, \ldots, n, \ldots$ converges to the limit $\infty : \mathbb{N}_\infty$.

2. A sequence $\mathbb{N} \to X$ converges to a limit $x_\infty : X$ if and only if it extends to a function

$$\mathbb{N}_\infty \to X$$

   that maps ∞ to x_∞.

$\mathbb{N}_\infty$ is the generic convergent sequence.

Also called the one-point compactification of $\mathbb{N}$.

The type $\mathbb{N}_\infty$

Taken to be that of decreasing binary sequences.

1. We imagine $1^n 0^\omega \longrightarrow 1^\omega$.
2. We notationally identify

$$\begin{aligned} n &\sim 1^n 0^\omega, \\ \infty &\sim 1^\omega. \end{aligned}$$

3. In Martin-Löf type theory, subtypes are defined using $\sum$:

$$\mathbb{N}_\infty = \sum_{\alpha \colon 2^{\mathbb{N}}} \prod_{n \colon \mathbb{N}} (\alpha_n = 0 \to \alpha_{n+1} = 0).$$

Intrinsic convergence

Definition. A sequence $x\colon \mathbb{N} \to X$ *intrinsically converges* to a limit $x_\infty\colon X$ if and only if it extends to a function

$$\hat{x}\colon \mathbb{N}_\infty \to X$$

that maps ∞ to x_∞.

The topological-topos interpretation of Martin-Löf theory makes this good.

This is just encouragement: We consider this definition independently of any model.

Any function of any two types is sequentially continuous

Without invoking Brouwerian continuity axioms.

Proof. Automatic.

Let $f\colon X \to Y$ be a function.

If $x_n \to x_\infty$ in X, then there is $\mathbb{N}_\infty \to X$ that maps n to x_n and ∞ to x_∞.

Now compose this function $\mathbb{N}_\infty \to X$ with your given function $f\colon X \to Y$.

The resulting function $\mathbb{N}_\infty \to Y$ maps n to $f(x_n)$ and ∞ to $f(x_\infty)$.

This means that $f(x_n) \to f(x_\infty)$.

Q.E.D.

Sensible to consider convergence up to isomorphism in U

Definition. We say that a sequence of types

$$X : \mathbb{N} \to U$$

intrinsically converges to a limit $X_\infty : U$ if and only if it extends to a function

$$\hat{X} : \mathbb{N}_\infty \to U$$

with

$$\hat{X}_n \cong X_n, \qquad \hat{X}_\infty \cong X_\infty.$$

For a univalent universe, this definition is equivalent to the previous one.

Because type isomorphism is equivalent to type equality in such a universe.

Theorem. The universe is intrinsically indiscrete

Every sequence of types converges to any type.

Assuming the axiom of function extensionality.

Formalized in Agda, but I want to give you an informal, rigorous proof here.

`www.cs.bham.ac.uk/~mhe/agda/TheTopologyOfTheUniverse.html`

Lemma. Every sequence of types converges to the type $\mathbf{1}$

Given $X : \mathbb{N} \to U$, define $\hat{X} : \mathbb{N}_\infty \to U$ by

$$\hat{X}_u = \prod_{k\,:\,\mathbb{N}} (u = k \to X_k).$$

Then

$$\hat{X}_n = \prod_{k\,:\,\mathbb{N}} (n = k \to X_k) \cong X_n,$$

and

$$\hat{X}_\infty = \prod_{k\,:\,\mathbb{N}} (\infty = k \to X_k) \cong \prod_{k\,:\,\mathbb{N}} (\mathbf{0} \to X_k) \cong \prod_{k\,:\,\mathbb{N}} \mathbf{1} \cong \mathbf{1}.$$

This is actually rather non-trivial, requiring subtle identity-type inductions.

Theorem. Every sequence of types converges to any type

Let $X : \mathbb{N} \to U$ and $Y : U$ be given.

(i) $\boxed{X_n \longrightarrow \mathbf{1}}$ Previous lemma.

(ii) $\boxed{(\mathbf{0})_n \longrightarrow \mathbf{1}}$ Special case of (i).

(iii) $\boxed{(\mathbf{0})_n \longrightarrow Y}$ Multiply (ii) by Y, and use $\mathbf{0} \times Y \cong \mathbf{0}$ and $\mathbf{1} \times Y \cong Y$.

(iv) $\boxed{(\mathbf{1})_n \longrightarrow \mathbf{0}}$ Compose (ii) with $(- \to \mathbf{0})$ and use $(\mathbf{0} \to \mathbf{0}) \cong \mathbf{1}$ and $(\mathbf{1} \to \mathbf{0}) \cong \mathbf{0}$.

(v) $\boxed{X_n \longrightarrow \mathbf{0}}$ Multiply (i) and (iv), and use $X \times \mathbf{1} \cong X$ and $\mathbf{1} \times \mathbf{0} \cong \mathbf{0}$.

(vi) $\boxed{X_n \longrightarrow Y}$ Add (iii) and (v), and use $\mathbf{0} + X_n \cong X_n$ and $Y + \mathbf{0} \cong Y$.

Corollary. Rice's Theorem for the Universe

It is a taboo to say that the universe has a non-trivial, extensional, decidable property.

For any extensional $P: U \to 2$ and $X, Y : U$, if $P(X) \neq P(Y)$ then WLPO.

We say that $P: U \to 2$ is extensional if $X \cong Y \implies P(X) = P(Y)$.

(For a univalent universe, all decidable properties are extensional.)

WLPO is the statement that every binary sequence is constantly 1 or it isn't.

$$P\colon U \to 2 \textbf{ extensional}, \ X, Y : U, \quad P(X) \neq P(Y) \implies \textbf{WLPO}$$

Assume w.l.o.g. that $P(X) = 0$ and $P(Y) = 1$.

By the Universe Indiscreteness Theorem, there is $Q\colon \mathbb{N}_\infty \to U$ with

$$\forall n\colon \mathbb{N}(Q(n) \cong X), \qquad Q(\infty) \cong Y.$$

Let $p\colon \mathbb{N}_\infty \to 2$ be $P \circ Q$. By the extensionality of P, we have that

$$\forall n\colon \mathbb{N}(p(n) = 0), \qquad p(\infty) = 1.$$

Hence for any given $x\colon \mathbb{N}_\infty$ we can decide whether $x = \infty$ by checking the decidable condition

$$p(x) = 1.$$

This amounts to WLPO.

It is decidable whether every binary sequence is constanly 1.

Sharper version of Rice's Theorem for the Universe

Found by Alex Simpson after I gave a similar talk nine days ago at MFPS'2012.

T.F.A.E.

1. There is a non-trivial, extensional map $U \to 2$.
2. $\forall P : U(\neg P \vee \neg\neg P)$.

Alex's proof is nice and short, and doesn't rely on our proof of the Universe Indiscreteness Theorem.

But it does rely, like ours, on the axiom of function extensionality for small types.

Challenge

Reconcile the old and new topological views of types as spaces.

1. (Old) Types are spaces, functions are continuous.
2. (New) Types are homotopy types.

My feeling is that types are topological blah-blah-blah-groupoids.

1. As a space, $2^{\mathbb{N}}$ is rather rich, but as a groupoid it is trivial.
2. As a space, U is trivial, but as a groupoid it is rather rich.

Open problem. Come up with a model (e.g. simplicial objects in the topological topos) that simultaneously accounts for the crucial aspects of the old and new topological views of types in constructive mathematics.

Understand this in a model-independent way.

Summary

1. The universe is intrinsically indiscrete.

   Up to isomorphism, assuming extensionality.
   We only need to know that it has very basic closure properties.

2. Hence it satisfies the conclusion of Rice's Theorem.

   So there is no hope of extending the theory with elimination rules $U \to Y$ where Y is a small type, unless Y is indiscrete too.

   But notice that there are plenty of definable functions $U \to U$.

3. Thomas Streicher has looked at what happens in the models.

   In the topological topos, U only has constant (extensional or not) maps into 2.
   And into the Sierpinski space as well. (Indiscrete topological reflection.)
   We are joining these results in a single paper.