Compact, totally separated and well-ordered types in univalent mathematics

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Summary

Work in a spartan univalent type theory.

(Development in Agda at github (`martinescardo/TypeTopology`).)

1. For a wide range of *infinite* types X, we have that for every $p : X \to \mathbb{2}$, the type

   $$\Sigma(x : X), p\,x = 0$$

   of roots of p is decidable, where $\mathbb{2} := \mathbb{1} + \mathbb{1}$ and $0, 1$ are its two points.

   We can either find a root of p or else tell that there is none.

2. The simplest counter-example is the type of natural numbers.

   (When $X = \mathbb{N}$, this is Bishop's Limited Principle of Omniscience (LPO).)

   The simplest example is the the type of conatural numbers.

3. The examples here turn out to be ordinals, and maybe this is not a coincidence.

Our univalent type theory

A spartan Martin-Löf type theory.

$\mathbb{0}$, $\mathbb{1}$, $\mathbb{N}$, $+$, Π, Σ, W, Id.

Hierarchy of open-ended universes, ranged over by $\mathcal{U}, \mathcal{V}$.

Intensional.

η-rules for Π, Σ, W.

Axioms.

Existence of propositional truncations.

Univalence.

(Now we could use cubical type theory and cubical Agda instead of axioms.)

Compact types

We consider three notions of exhaustively searchable type.

- We say that a type X is *compact* if the type $\Sigma(x : X), p\,x = 0$ is decidable for every $p : X \to \mathbb{2}$. (It is decidable whether p has a root.)

(Sometimes Σ-*compact* for emphasis.)

We also consider two successively weaker notions, namely

- $\exists$-*compactness* (it is decidable whether there is an unspecified root) and
- Π-*compactness* (it is decidable whether all points are roots),

obtained by replacing Σ by $\exists$ and Π in the definition of compactness.

E.g. for $X :\equiv \mathbb{N}$, we have that Σ- and $\exists$-compactness agree and amount to LPO.

Π-compactness amounts to WLPO.

Justification of the topological terminology

For the model of simple types consisting of Kleene–Kreisel spaces, these notions of compactness agree and coincide with topological compactness under classical logic.

But we reason constructively here.

Ordinals

An ordinal is a type X equipped with a proposition-valued binary relation $- < - : X \to X \to \mathcal{U}$ which is

- transitive,
- well-founded (satisfies transfinite induction), and
- extensional (any two elements with the same predecessors are equal).

The HoTT Book additionally requires the type X to be a set, but we show that this follows automatically from extensionality.

E.g. the types of natural and conatural numbers are ordinals.

By univalence, the type of ordinals in a universe is itself an ordinal in the next universe, and in particular is a set.

Ordinal arithmetic

- Addition is implemented by the type former $+$, and
- multiplication by the type former $\times$ with the lexicographic order.

The compact ordinals we construct are, moreover, *order-compact* in the sense that a minimal element of $\Sigma(x : X), p\,x = 0$ is found, or else we are told that this type is empty.

Additionally, we have a selection function of type $(X \to \mathbb{2}) \to X$ which gives the infimum of the set of roots of any $p : X \to \mathbb{2}$.

In particular our compact ordinals have a top element by considering $p = \lambda x.1$.

Discrete types

We say that a type is discrete if it has decidable equality.

Again this corresponds to the topological notion with the same name.

Totally separated type

It may happen that a non-trivial type has no nonconstant function into the type $\mathbb{2}$ of booleans so that it is trivially compact.

We again borrow terminology from topology (for spaces whose clopens separate the points).

We say that a type is totally separated if the functions into the booleans separate the points, in the sense that

- any two points that satisfy the same boolean-valued predicates are equal.

This is a boolean Leibnez principle.

Totally separated reflection

- We construct a totally separated reflection for any type, and
- show that a type is compact, in any of the three senses, if and only if its totally separated reflection is compact in the same sense.

Interplay between the notions

We show that compact types, totally separated types, discrete types and function types interact in very much the same way as their topological counterparts, where

- arbitrary functions in type theory play the role of continuous maps in topology, and
- *without* assuming Brouwerian continuity axioms.

For instance,

1. if the types $X \to Y$ and Y are discrete then X is Π-compact, and
2. if $X \to Y$ is Π-compact, and X is totally separated and Y is discrete, then X is discrete, too.
3. The simple types are all totally separated, which agrees with the situation with Kleene–Kreisel spaces, but
4. it is easy to construct types which fail to be totally separated,
   - e.g. the homotopical circle,

   or whose total separatedness gives a constructive taboo
   - e.g. $\Sigma(x : \mathbb{N}_\infty), x = \infty \to \mathbb{2}$, where we get two copies of the point ∞.

Notation for discrete and compact ordinals

We define infinitary ordinal codes, or expression trees, similar to the so-called Brouwer ordinals, including

- one,
- addition,
- multiplication, and
- countable sum with an added top point.

Two Interpretations of the notation

We interpret these trees in two ways, getting

- discrete and
- compact ordinals respectively.

In both cases, addition and multiplication nodes are interpreted as ordinal addition and multiplication.

But in the countable sum with a top point, the top point is added with

- $- + \mathbb{1}$ in one case, and so is isolated, and
- by a limit-point construction in the other case.

  (Given our sequence $\mathbb{N} \to \mathcal{U}$ of types, we extend it to a family $\mathbb{N}_\infty \to \mathcal{U}$ so that it maps ∞ to a singleton type, by a certain universe injectivity construction, and then take its sum).

Two interpretations of ordinal notations

We denote the above interpretations of ordinal notations ν by

- Δ_ν,
- K_ν.

We have that

1. Δ_ν is discrete and a retract of $\mathbb{N}$.
2. K_ν is compact, totally separated and a retract of $\mathbb{N} \to \mathbb{2}$.
3. $\Delta_\nu \hookrightarrow K_\nu$.

   E.g. $\mathbb{N} + 1 \hookrightarrow \mathbb{N}_\infty$.
4. This embedding is a bijection $\iff$ LPO holds.
5. But it always has empty complement.
6. It is order preserving and reflecting.
7. By transfinite iteration of the countable sum, one can get rather large compact ordinals using Setzer's work.