Compact ordinals in HoTT/UF

Martín Escardó
School of Computer Science
University of Birmingham, UK

CIRM
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The problem addressed here

1. Suppose every inhabited subset of $\mathbb{N}$ has a least element.
   Then $p \vee \neg p$ for every proposition p.
   Consider $\{x : \mathbb{N} \mid (p \wedge x=0) \vee (x>0)\}$

2. Similarly, if for any given (even decidable) subset of $\mathbb{N}$ we can either tell whether it is empty or else exhibit an element, then excluded middle follows.

3. However, we can exhibit plenty of "compact" ordinals such that for any given decidable subset, we can either tell it is empty or else exhibit a minimal element.

(we'll give our definition of ordinal later — it is the same as that of the HoTT book.)

simplified version of $\boxed{\text{The problem addressed here}}$

$$2 = \{0,1\} \cong 1+1 \neq \Omega$$

Given a set X and $p : X \to 2$,

- either exhibit $x \in X$ such that $p(x) = 0$ (a root of p)
- or else determine that p has no root.

$\boxed{\text{For which sets } X \text{ can this be done?}}$

- In terms of computation, this is an exhaustive search problem
- In terms of logic, this is a choice problem.
- In terms of topology, this turns out to be a compactness problem.

Can we exhaustively search an infinite set mechanically?

can we prove non-trivial instances of choice?

Our type theory

For some results we generalize to inductive-recursive types

Martin-Löf Type Theory

MLTT $\mathbb{0}, \mathbb{1}, \mathbb{N}, +, \times, \Sigma, \Pi, \mathrm{Id}, \mathcal{U}, W$

\+

univalence (so in particular we have functional and propositional extensionality)

\+

set quotients ($\iff$ propositional truncations + set replacement)

Many models

We reason constructively, so:
Our results hold in all models.

- Types are sets.
- Types are spaces.
- Types are homotopy types.
- Types are "sets with computational structure" (realizability).
- Types are the objects of a topos.

One particular model plays a guiding role

concrete sheaves

general sheaves

Topological topos $\Omega, \forall, \exists$

Kuratowski limit spaces Π, Σ

Sequential topological spaces

$\mathbb{0}, \mathbb{1}, \mathbb{N}, +, \times, \to$

site

$\mathbb{N}_\infty$ cfs

canonical coverage

Johnstone 1979

Examples of MLTT definable objects in that topos

- $\mathbb{N}$ and $\mathbb{2} \stackrel{\text{def}}{=} \mathbb{1}+\mathbb{1}$ get the discrete topology.

- $\mathbb{N} \to \mathbb{2}$ is the Cantor space, and $\mathbb{N} \to \mathbb{N}$ is the Baire space.

- $\mathbb{N}_\infty \stackrel{\text{def}}{=} \sum_{\alpha : \mathbb{N} \to \mathbb{2}} \prod_{i : \mathbb{N}} \alpha_i \geq \alpha_{i+1}$ is the one-point compactification of $\mathbb{N}$.

$\underline{n} \stackrel{\text{def}}{=} 1^n 0^\omega$ $\qquad$ $\mathbb{N} \hookrightarrow \mathbb{N}_\infty$

$\infty \stackrel{\text{def}}{=} 1^\omega$ $\qquad$ $n \longmapsto \underline{n}$

- $\sum_{x : \mathbb{N}_\infty} ((x = \infty) \to \mathbb{2})$ looks like this

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \quad \infty_0 \quad \infty_1$

This is compact T_1 but not Hausdorff.

We have $\{0, 1, \ldots, \infty_0\} \cap \{0, 1, \ldots, \infty_1\} = \mathbb{N}$

compact

not compact

Mathematical expression of the problem in our system

We can pick a root of p if it has any.

$$\prod p : X \to 2, \left(\sum x : X,\ p\,x = 0\right) + \underbrace{\left(\prod x : X,\ p\,x = 1\right)}_{\neg \sum x : X,\ p\,x = 0}$$

- Stronger than excluded middle.
- We are making a choice.

We have Σ rather than $\exists$.

We ask which types X satisfy this choice principle.

Definition. We call such types compact.

$\boxed{\text{All types are compact} \iff \text{global choice holds}}$

Global choice: We can choose a point of every non-empty type

$$\prod_{X:\mathcal{U}} \underbrace{\neg\neg X}_{X \text{ is non-empty}} \to X$$

E.g. Voevodsky's model of simplicial sets

- stronger than choice, which is consistent with univalence.
- Contradicts univalence.
- But there are plenty of compact types in HoTT/UF.
- The ones we can construct are all equipped with well-orders.

Ordinals

A type X equipped with a proposition-valued relation $<$ s.t.

1. $<$ is transitive.
2. If two points have the same predecessors, then they are equal.
3. $<$ satisfies transfinite induction:

$$\left(\prod x : X \left(\prod y : X,\ y < x \to P y\right) \to P x\right) \to \prod x : X,\ P x$$

- X is automatically a set by (2) (its identity types are propositions)
- Trichotomy $x < y$ or $x = y$ or $x > y$ for all ordinals is equivalent to LEM.
- But there are plenty of trichotomous ordinals without assuming LEM.

The large type of all small ordinals

Univalence implies that this type

1. is itself a large ordinal,
2. has suprema of small-indexed families.

(we'll discuss this further later)

Functions $p : X \to 2$

They classify complemented subtypes of X.

$$X \simeq \left(\sum_{x:X} p\,x = 0\right) + \left(\sum_{x:X} p\,x = 1\right).$$

$$\begin{tikzcd} Y \arrow[r] \arrow[d, hook, "\text{Complemented}"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{1} \arrow[d, "0"] \\ X \arrow[r, "p"'] & 2 \end{tikzcd}$$

- Topological topos. They classify clopen subspaces.

Totally separated types

Recall

Definition. A type X is called *compact* if

$$\prod p : X \to 2,\ \left(\sum x : X,\ p x = 0\right) + \left(\prod x : X,\ p x = 1\right).$$

This definition is not good unless there are plenty of maps $X \to 2$.

Definition. A type X is called *totally separated* if

$$\prod x, y : X,\ \left(\prod p : X \to 2,\ p x = p y\right) \to x = y.$$

In the topological topos. The clopens separate the points.

(Topological notion with the same name.)

Some facts

1. Totally separated types are sets (their identity types are propositions)

2. They form an exponential ideal (more generally a "Π-ideal") and are closed under $+$, $\times$, retracts and include $\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbb{N}, \mathbb{N}_\infty$ and all discrete types (those with decidable equality).

3. They are not closed under Σ in general.

not totally separated

Example. In the topological topos, the type $\to$ $\underbrace{\sum_{x:\mathbb{N}_\infty}}_{\text{totally separated}} \big((x=\infty) \to \mathbf{2}\big)$ is not totally separated.

totally separated

(Compact totally separated spaces are Hausdorff. Also known as Stone spaces.)

4. Define the simple types to be the smallest collection of types including $\mathbb{0}, \mathbb{1}, \mathbb{N}$ and closed under $\times, +, \to$.

The simple types are all totally separated (by (2) above).

5. In the topological topos, a subtype of a simple type is compact in the above type-theoretic sense iff it is compact in the topological sense.

In this case the inclusion is a section and hence the subtype is itself totally separated.

6. Every type X has a totally separated reflection, the image of $X \to ((X \to 2) \to 2)$.

$x \mapsto \lambda p.\, p\, x$

Counter-example to compactness

A so-called constructive taboo.

- The set $\mathbb{N}$ of natural numbers fails to be compact
- The compactness of $\mathbb{N}$ amounts to Bishop's LPO (Limited Principle of Omniscience).

More precisely, LPO is independent of MLTT

- False in realizability models (not computable)
  in topological models (not continuous)
- True in the model of classical sets (by choice)

Probably the simplest infinite example

$$\mathbb{N}_\infty := \Sigma\, \alpha : 2^{\mathbb{N}},\ \Pi\, i : \mathbb{N},\ \alpha_i \geq \alpha_{i+1}$$

That is, the type of decreasing binary sequences.

$$\underline{n} := 1^n 0^\omega$$

$$\infty := 1^\omega$$

We have an injection
$$\begin{aligned} \mathbb{N} &\longrightarrow \mathbb{N}_\infty \\ n &\longmapsto \underline{n} \end{aligned}$$

Theorem of HoTT/UF

The type $\mathbb{N}_\infty$ is compact.

(JSL '2013)

↳ Done in a weaker system (Gödel's system T)

Proof sketch (with the difficult part omitted)

- Given $p : \mathbb{N}_\infty \to 2$, (<u>not</u> assumed be continuous)

  define $\beta_n = \min(p\,\underline{0}, p\,\underline{1}, \ldots, p\,\underline{n})$ Formula for the infimum of the set of roots.

- This is clearly decreasing.

- Now we check whether $p\beta = 0$ or $p\beta = 1$.

  (0) If $p\beta = 0$ then we've found a root.

  (1) If $p\beta = 1$ then $p\alpha = 1$ for all $\alpha : \mathbb{N}_\infty$ and so there is no root. (This is easy classically and less so constructively.)

In the pub $\mathbb{N}_\infty$ there is a person $\beta : \mathbb{N}_\infty$ such that if β drinks, then everybody drinks.

Some consequences (decision procedures)

(1) For every $p : \mathbb{N}_\infty \to 2$ either $\Pi n : \mathbb{N}, p\, \underline{n} = 1$ or $\neg \Pi n : \mathbb{N}, p\, \underline{n} = 1$

(JSL'2013)

quantification over the natural numbers! Not over $\mathbb{N}_\infty$.

(2) Given $f : \mathbb{N}_\infty \to \mathbb{N}$, we can decide whether it is <u>not</u> continuous.

(3) There is some discontinuous $f : \mathbb{N}_\infty \to \mathbb{N}$ iff WLPO holds.

(Bishop's principle of Weak Limited omniscience, $\Pi p : \mathbb{N} \to 2, (\Pi n. p n = 1) + \neg(\Pi\, p n = 1)$ which is also independent of MLTT)

(MSCS'2015)

Some applications of the compactness of $\mathbb{N}_\infty$

1. Pierre Pradic & Chad E. Brown. Arxiv '2019

   Cantor-Bernstein implies excluded middle

   arxiv 1904.09193

   (Also implemented in Coq.)

   (and so is equivalent to excluded middle)

2. Dag Normann & William Tait. Springer '2017

   On the computability of the Fan Functional

   (They use the system T compactness of $\mathbb{N}_\infty$ to fill a gap in an unpublished but widely circulated 1958 manuscript by Tait.)

Compact sets in our type theory

(1) $\mathbb{0}$, $\mathbb{1}$ and $\mathbb{N}_\infty$ are compact.

(2) If X and Y are compact then so are $X+Y$ and $\underbrace{X \times Y}^{\text{Baby Tychonoff.}}$.

(3) If X is a compact set and A is a family of compact sets indexed by X, then its disjoint union $\sum_{x:X} A\,x$ is a compact set.

(4) If furthermore (a) we have a function that picks an element of $A\,x$ for any given $x:X$, and
(b) the set X has at most one element,
then the cartesian product $\prod_{x:X} A\,x$ is compact. Micro-Tychonoff.

Does arbitrary Tychonoff hold?

Is the Cantor type $\mathbb{N} \to 2$ (provably) compact in our type theory?

No.

- The compactness of $\mathbb{N} \to 2$ is independent.

True in the topological topos (notions of compactness coincide)

False in Hyland's effective topos (Kleene tree to blame)

(realizability topos over Kleene's K_1)

True in the Kleene-Vesley topos

(realizability over Kleene's K_2)

Perhaps amazingly, these two toposes have the same simple types.

(more precisely, the full subcategories on the objects that arise as the interpretation of the simple types are equivalent.)

Building more compact sets

- The compact sets that we have constructed so far are all well-ordered.

(1) $\mathbb{0}$

$\mathbb{1}$

$\mathbb{N}_\infty$

(2) $X + Y$

$X \times Y$

(3) $\Sigma\, x : X,\ A\, x$

lexicographic order

Problem with Σ

- But we can't get very high ordinals with just the above.
- This is what we address next

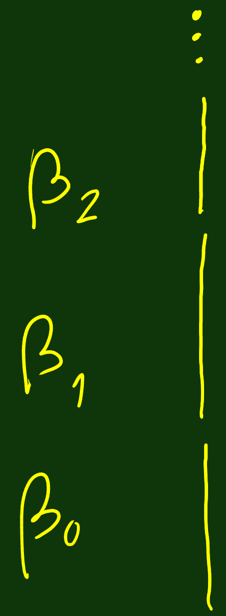
after we address this

Ordinal sums

$\alpha : \mathrm{Ord} \qquad \beta : \langle \alpha \rangle \to \mathrm{Ord}$

lexicographic order $(x,y) < (x',y') \overset{\text{def}}{=} (x < x') + \sum_{p : x = x'} \mathrm{transport}\ p\ y < y'$

E.g. $\alpha \overset{\text{def}}{=} \omega$



Problem with ordinal sums

The lexicographic order is not extensional in general

Can derive excluded middle

It is extensional if

This is very lucky because our compact ordinals do have top.

1) The given orders have top elements — then so does the sum

or 2) The given orders are trichotomous — then so is the sum

or 3) The given orders are cotransitive — but I have no reason to believe that so would be the sum

or 4) Excluded middle holds (by (2))

Suprema of ordinals

- Does every family $\alpha : I \to \mathrm{Ord}$ have a supremum constructively?
- As far as I know, this hadn't been answered before. Left open by Forsberg, Kraus & Xu (MFCS'21).
- Two answers, by Tom de Jong & myself independently. Both implemented in Agda by Tom.
- Mine is as follows. Define

$$\sum_{i:I} \langle \alpha_i \rangle \longrightarrow \mathrm{Ord}$$

$$(i, x) \longmapsto \alpha_i \downarrow x$$

Then the supremum is just the image of this function. Although Ord is large, the image is small (assuming quotients)

Suprema of families of ordinals

Define $\sum_{i:I} \langle \alpha_i \rangle \longrightarrow \mathrm{Ord}$

$(i, x) \longmapsto \alpha_i \downarrow x$

Then the <u>supremum</u> is just the <u>image</u> of this function.
Although Ord is large, the image is small (assuming quotients).

<u>Corollary</u> If I is compact and each α_i is compact, then so is the supremum of α_i.

<u>Because</u> $\sum_i \alpha_i$ is compact and images of compact types are compact. $\square$

We need a further ingredient

Extending families of compact sets

General situation:

$$
\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[dr, "A"'] & Y \arrow[d, dotted, "B"] \\
& \mathcal{U}
\end{tikzcd}
$$

embedding

type family

type universe

extension of A along j

Interested in:

$$
\begin{tikzcd}
\mathbb{N} \arrow[r, hook, "\text{canonical embedding}"] \arrow[dr, "A"'] & \mathbb{N}_\infty \arrow[d, dotted, "B"] \\
& \mathcal{U}
\end{tikzcd}
$$

Want: If $A\,x$ compact for every $x : X$, then $B\,y$ compact for every $y : Y$.

Because then: By (3), if Y is also compact, then $\Sigma\, y : Y,\ B\,y$ compact too.

$j^{-1}(y) = \sum_{x:X} jx = y$

Family extension problem

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "A"'] & Y \arrow[d, dotted, "B"] \\ & \mathcal{U} \end{tikzcd}$$

(MSCS'2021. "Injective types in univalent mathematics".)

This set has at most one element. (because j is an embedding)

Smallest solution (left kan extension): $B\,y := \sum_{(x,-):j^{-1}(y)} A\,x$

Largest solution (right kan extension): $B\,y := \prod_{(x,-):j^{-1}(y)} A\,x$

[It is this that works for the wish of the previous board.] why? By Micro-Tychonoff

Summary of the previous reasoning

$$\begin{tikzcd} X \arrow[r, hook, "j\ \text{given}"] \arrow[dr, "A"', "\text{given}"'] & Y \arrow[d, dotted] \\ & \mathcal{U} \end{tikzcd} \qquad B\,y := \prod_{x : j^{-1}(y)} A\,x$$

Special case of interest: $\begin{tikzcd} \mathbb{N} \arrow[r, hook] \arrow[dr, "A"'] & \mathbb{N}_\infty \arrow[d, dotted, "B"] \\ & \mathcal{U} \end{tikzcd}$

Theorem If the set $A\,x$ is compact for every $x : X$, then then set $B\,y$ is compact for every $y : Y$.

Corollary If additionally Y is compact, then so is $\sum_{y:Y} B\,y$.

$\boxed{\text{In the special case of interest we have } B(\infty) \simeq \mathbb{1}}$

$\boxed{\text{But this is not enough}}$

The above says that type universes $\mathcal{U}$ are injective

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "A"'] & Y \arrow[dl, dotted] \\ \mathcal{U} & \end{tikzcd}$$

$$\bar{A}(y) = \prod_{(x,-) : j^{-1}(y)} A\,x$$

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "\alpha"'] & Y \arrow[dl, dotted, "\bar{\alpha}"] \\ \mathrm{Ord}\,\mathcal{U} & \end{tikzcd}$$

We need to order $\bar{\alpha}(y)$.

We define, for $u, v : \langle \bar{\alpha}\, y \rangle$,

$$u < v = \sum_{\sigma : j^{-1}(y)} u\,\sigma < v\,\sigma$$

Theorem. The type of (topped) ordinals also is injective.

Proof That of the injectivity of a universe for + additional construction of the order + checking it works. □

$\boxed{\text{Next}}$ using the previous machinery

- Compact ordinals induced by Brouwer ordinal expressions.

- A generalization to a universe à la Tarski of compact ordinals.

Brouwer ordinal codes

They seem to be due to Hilbert!

A type B inductively defined by constructors

$$Z : B$$ "zero"

$$S : B \to B$$ "successor"

$$L : (\mathbb{N} \to B) \to B$$ "limit"

Four interpretations of Brouwer codes as ordinals

0)

$$[\![Z]\!] = \mathbb{0}$$
$$[\![S b]\!] = [\![b]\!] + \mathbb{1}$$
$$[\![L b]\!] = \sup_i [\![b_i]\!]$$

standard interpretation

1)

$$[\![Z]\!] = \mathbb{0}$$
$$[\![S b]\!] = [\![b]\!] + \mathbb{1}$$
$$[\![L b]\!] = \sum_i [\![b_i]\!]$$

trichotomous interpretation

2)

$$[\![Z]\!] = \mathbb{1}$$

! topped

$$[\![S b]\!] = [\![b]\!] + \mathbb{1}$$
$$[\![L b]\!] = \sup_{i : \mathbb{N}} \overline{[\![b_i]\!]}$$

compact interpretation

3)

$$[\![Z]\!] = \mathbb{1}$$

! topped

$$[\![S b]\!] = [\![b]\!] + \mathbb{1}$$
$$[\![L b]\!] = \sum_{i : \mathbb{N}} \overline{[\![b_i]\!]}$$

Overline means extension to $\mathbb{N}_\infty$ by injectivity.

Compact totally separated interpretation

$\boxed{\text{Assuming excluded middle}}$

which admittedly is not very useful for what we are investigating.

$$\begin{array}{lccccl}
\text{standard} & [\![b]\!]_{\sup} & \le & [\![b]\!]_{\Sigma} & & \text{trichotomous} \\
 & \rotatebox[origin=c]{-90}{$\le$} & & \rotatebox[origin=c]{-90}{$\le$} & & \\
\text{compact} & [\![b]\!]_{\overline{\sup}} & \le & [\![b]\!]_{\overline{\Sigma}} & & \text{compact totally separated}
\end{array}$$

In the next page we see what happens constructively here.

why do we need excluded middle?

Because $(-) + \mathbb{1}$ is monotone $\iff$ excluded middle holds.

Theorems

The ordinal

$[\![b]\!]_{\Sigma}$

- is discrete, in fact trichotomous
- is a retract of $\mathbb{N}$
- so countable
- Not compact unless LPO holds

The ordinal

$[\![b]\!]_{\bar{\Sigma}}$

- is compact
- is a retract of $\mathbb{N} \to 2$
- so totally separated
- Is not countable unless LPO holds
- Is not discrete unless LPO holds

Even better:

Every decidable subset is either empty or has a least element.

- There is an order-preserving-reflecting embedding

$$[\![b]\!]_{\Sigma} \hookrightarrow [\![b]\!]_{\bar{\Sigma}}$$

whose image has empty complement.

- LPO $\Rightarrow$ this embedding is a bijection $\Rightarrow$ WLPO.

In models:

The embedding doesn't have a computable / continuous inverse.

Illustration | The ordinal $\omega+1$.

$$n \longmapsto 1^n 0^\omega$$

$$\mathbb{N} + \mathbb{1} \hookrightarrow \mathbb{N}_\infty$$

$$* \longmapsto 1^\omega$$

- Discrete
- compact iff LPO
- Countable

- Compact
- discrete iff WLPO
- countable iff LPO

- bijection iff LPO,
- but its image has empty complement.

Every decreasing sequence is of one of the forms $1^n 0^\omega$ and 1^ω.

There is no decreasing sequence other than $1^n 0$ and 1^ω.

Universes à la Tarski of compact ordinals

We define $E : \mathcal{U}_0$

$\Delta : E \to \mathrm{Ord}^{\mathrm{Top}}$ trichotomous

by <u>induction-recursion</u>. Then we define, by <u>recursion</u>,

$K : E \to \mathrm{Ord}^{\mathrm{Top}}$ compact

(totally separated?)

We define E, using Δ, inductively by the following constructors:

$$\ulcorner \mathbb{1} \urcorner : E \qquad \_ \ulcorner + \urcorner \_ : E \to E \to E \qquad \ulcorner \Sigma \urcorner : (e : E) \to (\langle \Delta e \rangle \to E) \to E$$

$$\ulcorner \omega + 1 \urcorner : E \qquad \_ \ulcorner \times \urcorner \_ : E \to E \to E$$

$$E : \mathcal{U}_0$$

$$\Delta : E \to \mathrm{Ord}^{\mathrm{Top}}$$

We simultaneously define:

$$K : E \to \mathrm{Ord}^{\mathrm{Top}}$$

$$\iota : (e : E) \to \langle \Delta\, e \rangle \to \langle K\, e \rangle$$

$$\iota\text{-emb} : (e : E) \to \text{is-embedding}\,(\iota\, e)$$

$$\ulcorner \mathbb{1} \urcorner : E$$

$$\ulcorner \omega + 1 \urcorner : E$$

$$\_ \ulcorner + \urcorner \_ : E \to E \to E$$

$$\_ \ulcorner \times \urcorner \_ : E \to E \to E$$

$$\ulcorner \Sigma \urcorner : (e : E) \to (\langle \Delta\, e \rangle \to E) \to E$$

$\Delta\, \ulcorner \text{brocoli} \urcorner = \text{brocoli}$ recursively.

K is defined as Δ except that rather than

$$\Delta(\ulcorner \Sigma \urcorner\, e\, A) = \sum_{x : \langle \Delta e \rangle} \Delta(A\, x)$$

We define

$$K(\ulcorner \Sigma \urcorner\, e\, A) = \sum_{x : \langle K e \rangle} K(\overline{A}\, x)$$

what is $\overline{A}$?

$$
\begin{tikzcd}
\langle \Delta e \rangle \arrow[r, hook, "\iota e"] \arrow[d, "A"'] \arrow[dr] & \langle K e \rangle \arrow[d, dashed, "\bar{A}"] \\
E \arrow[r, "K"'] & \mathrm{Ord}^{\mathrm{Top}}
\end{tikzcd}
$$

Answer: extension by injectivity. (we need right kan)

We omit the definition of ι and the proof that it is an embedding.

The End