Continuity in constructive dependent type theory

Martín Hötzel Escardó

University of Birmingham, UK

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Brouwer's continuity principle

The value $f(\alpha)$ of a function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ depends only on a finite prefix of the sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$.

NB. This is continuity in the topological sense if we endow $\mathbb{N}$ with the discrete topology and $\mathbb{N}^{\mathbb{N}}$ with the product (=exponential) topology.

Question

How should one formulate Brouwer's continuity principle for functions

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in (intensional or extensional) Martin-Löf Type Theory?

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1. This question turns out to be subtler than it may seem at first sight.
   Even in the absence of function extensionality.
2. We of course don't expect the continuity principle to be provable.
3. But much less we expect it be disprovable.
4. However, perhaps surprisingly, its Curry–Howard interpretation actually is disprovable.
5. What does that mean, and what is the correct formulation of the continuity principle in MLTT?

Brouwer's continuity principle in predicate logic

$$\forall(f : \mathbb{N}^\mathbb{N} \to \mathbb{N}).\ \forall(\alpha : \mathbb{N}^\mathbb{N}).\ \exists(n : \mathbb{N}).\ \forall(\beta : \mathbb{N}^\mathbb{N}).\ \alpha =_n \beta \to f\alpha = f\beta.$$

Not provable in e.g. higher-type Heyting arithmetic (HA^ω).

But validated e.g. by realizability over Kleene's K_2 and by Johnstone's topological topos, among other well-known models.

Explain the topological topos a little bit in the board.

Brouwer's continuity principle in dependent type theory

Take the Curry–Howard interpretation of the above:

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta.$$

1. This implies $0 = 1$.

   This implication is a theorem of intensional Martin-Löf type theory, by adaptation of an old argument due to Kreisel, originally relying on extensionality.

2. Maybe shocking at first sight, but makes perfect topological sense.

   The above says explicitly that every f is continuous.

   But it also says implicitly that we can continuously find a modulus of continuity n of f at α as a function of f and α.

   It is the second, implicit continuity requirement that cannot hold.

Brouwer's continuity principle in dependent type theory

How do we formulate it in a consistent, and meaningful, way?

Brouwer's continuity principle in dependent type theory

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\|\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta\|.$$

1. $\|X\|$ = quotient of the type X by the chaotic equivalence relation.

   Definable as a large type as $\Pi(P : U).\,\mathrm{isProp}(P) \to (X \to P) \to P$.

   Here a type is a proposition if it has at most one element.

   Discuss this in the board.

2. Validated by the topological topos and some realizability toposes.

   In a topos, $\|X\|$ is the image of the unique map $X \to 1$.

   It is the truth value of the inhabitedness of X,
   without necessarily revealing an inhabitant.

3. We have $\|\Sigma(x : X).A(x)\| = (\exists(x : X).A(x))$ in any topos.

Uniform continuity

Joint work with Chuangjie Xu.

$\forall(f\colon 2^{\mathbb{N}} \to \mathbb{N}).\ \exists(n\colon \mathbb{N}).\ \forall(\alpha,\beta\colon 2^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta.$

1. Again not provable but consistent in HA^ω.
2. This time, its Curry–Howard interpretation

   $\Pi(f\colon 2^{\mathbb{N}} \to \mathbb{N}).\ \Sigma(n\colon \mathbb{N}).\ \Pi(\alpha,\beta\colon 2^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta$

   is also consistent.
3. Moreover, it is logically equivalent to

   $\Pi(f\colon 2^{\mathbb{N}} \to \mathbb{N}).\ \|\Sigma(n\colon \mathbb{N}).\ \Pi(\alpha,\beta\colon 2^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta\|,$

   assuming function extensionality.
4. Chuangjie has also constructively developed a variation of the topological topos modelling this, and implemented it in Agda.

Summary of claims

1. Continuity is not provable in HA^{ω}, but is validated in some models.
2. The Curry–Howard interpretation of continuity is always false.
3. Consistent type-theoretic formulation via propositional truncation.
4. For uniform continuity, it doesn't make any difference whether we truncate Σ or not.

Failure of the Curry–Howard interpretation of continuity

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta.$$

1. The above axiom talks explicitly about functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ only.

   But it implicitly makes an assertion about *all* functions $X \to Y$.

2. If we have a "probe" $\mathbb{N}^{\mathbb{N}} \to X$ and an "observation" $Y \to \mathbb{N}$, then the composite $\mathbb{N}^{\mathbb{N}} \to X \to Y \to \mathbb{N}$ of the three functions has to be continuous according to the above axiom.

   Any function $X \to Y$ of any two types becomes empirically continuous by probing X and observing Y.

   A remark is that in the model of Kleene–Kreisel continuous functionals, empirical continuity agrees with topological continuity.

   This remark is important for the intuition that guides the proof, but it doesn't feature in the proof, at least not explicitly.

Failure of the Curry–Howard interpretation of continuity

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta.$$

1. The above axiom talks explicitly about functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ only.

   But it implicitly makes an assertion about *all* functions $X \to Y$.

2. Any function $X \to Y$ of any two types becomes continuous by probing X with a function $\mathbb{N}^{\mathbb{N}} \to X$ and observing Y with a function $Y \to \mathbb{N}$.
3. By projection, the axiom gives a functional

   $$M : (\mathbb{N}^{\mathbb{N}} \to \mathbb{N}) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$$

   that assigns a modulus $n = M(f, \alpha)$ to f at the point α.

   Trouble: While all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ may be continuous, there can't be any continuous modulus-of-continuity functional M.

Proof of $0 = 1$

We set up an experiment to test the continuity of M.

1. Write $M(f) = M(f, 0^\omega)$ for the sake of brevity.

   0^ω is the infinite sequence of zeros, i.e. $\lambda i.0$.

   $0^n k^\omega$ consists of n zeros followed by infinitely many k's.

2. Let $m = M(\lambda\alpha.0)$.

   Define $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ to be $f(\beta) = M(\lambda\alpha.\beta(\alpha(m)))$, by probing M.

3. By expanding the definitions (which involves the ξ-rule), we get

$$f(0^\omega) = M(\lambda\alpha.0^\omega(\alpha(m))) = M(\lambda\alpha.0) = m,$$

   and hence

$$\Pi(\beta\colon \mathbb{N}^\mathbb{N}).0^\omega =_{Mf} \beta \to m = f\beta.$$

   For any $\beta\colon \mathbb{N}^\mathbb{N}$, by the continuity of $\lambda\alpha.\beta(\alpha m)$, we get $\Pi(\alpha\colon \mathbb{N}^\mathbb{N}).0^\omega =_{f\beta} \alpha \to \beta 0 = \beta(\alpha m)$.

4. Choosing $\beta = 0^{Mf+1}1^\omega$, we get $0^\omega =_{Mf+1} \beta$, and so $0^\omega =_{Mf} \beta$, and hence $f(\beta) = m$ and $\Pi(\alpha\colon \mathbb{N}^\mathbb{N}).0^\omega =_m \alpha \to \beta 0 = \beta(\alpha m)$.

5. Choosing $\alpha = 0^m(Mf+1)^\omega$, we have $0^\omega =_m \alpha$, and therefore $0 = \beta 0 = \beta(\alpha m) = \beta(Mf+1) = 1$. QED

Discussion

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta.$$

1. The problem with this formulation of the continuity axiom is the dependency of n on f and α, which is itself (empirically) continuous.

   This formulation of the axiom is saying more than we intended to say.

2. We have to break the implicit continuous dependency of the output n on the inputs f and α.

   A crude way to achieve this is to double-negate the conclusion:

   $$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\neg\neg\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta.$$

   But this is too weak.

The correct formulation of the continuity axiom should be

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\|\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta\|.$$

1. The axiom of choice is

   $$(\Pi(x : X).\|\Sigma(y : Y).A(x,y)\|) \to \|\Sigma(f : X \to Y).\Pi(x : X).A(x, f(x))\|.$$

2. Choice implies WLPO.

   (And even excluded middle if quotients are added to MLTT.)

Continuity implies $\neg\mathrm{WLPO}$.

Hence choice and continuity are together impossible.

Extensionality considerations play no role in this argument.

We now discuss uniform continuity

I will use the board.

1. Discussion in type theory.
2. A constructively developed variation of the topological topos.

And now I want to discuss continuity on $\mathbb{N}_\infty$

The one-point compactification of the natural numbers.

Still in the board.

If there is time left. Maybe wishful thinking.

Concluding summary and discussion

Summary:

1. Continuity is not provable in HA^{ω}, but is validated in some models.
2. The Curry–Howard interpretation of continuity is always false.
3. Correct type-theoretic formulation via propositional truncation.
4. For uniform continuity, it doesn't make any difference whether we truncate Σ or not.

Discussion:

1. Did Brouwer really mean the BHK interpretation?

   Should we call it the HK interpretation instead?

2. What is, should be, or can be constructive existence?

Some references related to continuity in type theory

1. Infinite sets that satisfy the principle of omniscience in any variety of constructive mathematics. JSL, 2013.

2. Constructive decidability of classical continuity. MSCS, 2014.

3. The inconsistency of a Brouwerian continuity principle with the Curry-Howard interpretation. TLCA, 2015, with Chuangjie Xu.

4. A constructive manifestation of the Kleene-Kreisel continuous functionals. Accepted for APAL, with Chuangjie Xu.

5. The universe is indiscrete. Accepted for APAL, with Thomas Streicher.

Old material follows, in case I need it

(From my IHP talk in June last year.)

A wish that can't be fulfilled literally

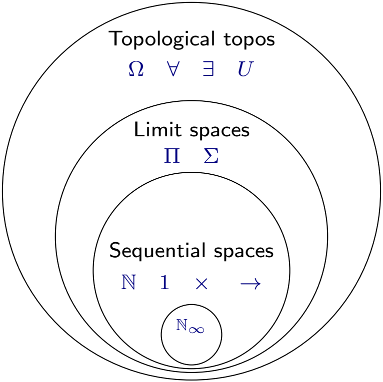
1. Types are interpreted as topological spaces.
2. Terms are interpreted as points of spaces.
3. Functions are interpreted as continuous maps.

The category of continuous maps of topological spaces is not even cartesian closed (it doesn't have exponentials (function spaces)).

Hence it can't interpret Gödel's system T or Martin-Löf type theory.

However, there are natural continuous models of type theory.

Johnstone's topological topos (1979)



1. The site is the category of continuous endomaps of the one-point compactification $\mathbb{N}_\infty$ of $\mathbb{N}$ with the canonical coverage.
2. Taking colimits of $\mathbb{N}_\infty$ in topological spaces gives sequential spaces.
3. The limit spaces arise as the subobjects of sequential spaces.

Examples of MLTT-definable objects of the topos

1. The interpretation of the type $\mathbb{N} \to 2$ gives the Cantor space $2^{\mathbb{N}}$.
2. The interpretation of the type $\mathbb{N} \to \mathbb{N}$ gives the Baire space $\mathbb{N}^{\mathbb{N}}$.
3. The interpretation of the simple types gives the Kleene–Kreisel continuous functionals. (Start from $\mathbb{N}$ and close under $\to$.)
4. The interpretation of the type

   $$\mathbb{N}_\infty \stackrel{\text{def}}{=} \left(\sum_{\alpha:\mathbb{N}\to 2} \prod_{n:\mathbb{N}} \alpha_n = 0 \to \alpha_{n+1} = 0 \right)$$

   gives the one-point compactification of $\mathbb{N}$, with $\infty \stackrel{\text{def}}{=} (\lambda i.1, -)$.

   Here "$=$" is the identity type, interpreted as an equalizer.
5. The interpretation of the type

   $$\sum_{x:\mathbb{N}_\infty} 2^{x=\infty}$$

   is a T_1, non-Hausdorff, but compact, space with two points at infinity,

   $$\infty_0 \stackrel{\text{def}}{=} (\infty, \lambda p.0), \quad \infty_1 \stackrel{\text{def}}{=} (\infty, \lambda p.1).$$

The topological topos validates continuity axioms

Continuity axiom (Cont)

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

$$\forall f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \forall \alpha : \mathbb{N}^{\mathbb{N}}.\ \exists n : \mathbb{N}.\ \forall \beta : \mathbb{N}^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Uniform continuity axiom (UC)

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

$$\forall f : 2^{\mathbb{N}} \to \mathbb{N}.\ \exists n : \mathbb{N}.\ \forall \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

- This assumes a classical meta-theory.
- Towards the end I discuss another topological topos developed within a constructive meta-theory by Chuangjie Xu and myself.

  (Also formalized in Agda by Chuangjie.)
- For the moment ignore constructivity issues until further notice.

Does the Brouwer-Heyting-Kolmogorov-Curry-Howard interpretation work too?

The topological topos is a lccc — it has Π and Σ.

If we apply the BHKCH interpretation:

Continuity axiom (Cont):

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Continuity axiom (Cont): ✕ Moreover, no topos can validate this.

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$\Pi f : 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n : \mathbb{N}.\ \Pi\alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$

Theorem of intensional Martin-Löf type theory

If all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous then $0 = 1$.

$$\left(\prod_{f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \prod_{\alpha : \mathbb{N}^{\mathbb{N}}} \sum_{n : \mathbb{N}} \prod_{\beta : \mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta \right) \to 0 = 1.$$

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I could instead say "not all functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous", but:

1. This would give the false impression that there might exist a non-continuous function to be found by looking hard enough.
   (In the topological topos all functions are continuous, and yet this holds.)

2. It is $0 = 1$ that our proof actually does give from the assumption.
   (A technicality that leads to the next item.)

3. We would need a universe to map the type $0 = 1$ to the type $\emptyset$, and our proof doesn't require universes.
   (So we are more general.)

Theorem of intensional Martin-Löf type theory

$$\left(\prod_{f\,:\,\mathbb{N}^{\mathbb{N}}\to\mathbb{N}}\ \prod_{\alpha:\mathbb{N}^{\mathbb{N}}}\ \sum_{n:\mathbb{N}}\ \prod_{\beta:\mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta\right) \to 0 = 1.$$

Proof sketch. Let

$$\phi : \prod_{f\,:\,\mathbb{N}^{\mathbb{N}}\to\mathbb{N}}\ \prod_{\alpha:\mathbb{N}^{\mathbb{N}}}\ \sum_{n:\mathbb{N}}\ \prod_{\beta:\mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta.$$

Using ϕ and the projections and choosing $\alpha = 0^\omega$, we get

$$M : (\mathbb{N}^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$$

and

$$\gamma : \prod_{f\,:\,\mathbb{N}^{\mathbb{N}}\to\mathbb{N}}\ \prod_{\beta:\mathbb{N}^{\mathbb{N}}} 0^\omega =_{Mf} \beta \to f0^\omega = f\beta.$$

Now define $m = M(\lambda\alpha.0)$ and consider

$$f\beta = M(\lambda\alpha.\beta(\alpha_m)).$$

Then argue $Mf = 0$ and $Mf > 0$ give $0 = 1$, using $f0^\omega = m$. (Induction on Mf not needed). Q.E.D.

Proof discussion

This is an adaptation of a well known argument (due to Kreisel?).

1. Continuity, choice and extensionality are together impossible.
2. No *extensional* modulus-of-continuity functional M.
3. But here we are working in *intensional* Martin-Löf type theory.
4. No *continuous* modulus-of-continuity functional M.
5. We used our hypothetical M to define a non-continuous function f and hence prove M wrong.
6. And this is exactly what is happening in the topological topos:
    - All functions are continuous.
    - But there is no continuous way of finding moduli of continuity.
    - No finite amount of information about f suffices to determine its modulus.

Σ versus $\exists$

Fix an object X.

1. Σ is understood in slices $\mathcal{E}/X$.
   If we have an object classifier U (universe), we can understand it as

   $$\Sigma : (X \to U) \to U.$$

   Given a family of objects we get an object.
2. $\exists$ is understood as a function

   $$\exists : (X \to \Omega) \to \Omega.$$

3. They are related via a reflection of U into Ω:

   $$U \underset{\hookleftarrow}{\overset{\|-\|}{\longrightarrow}} \Omega.$$

$$(\exists x : X.P(x)) = \|\Sigma x : X.P(x)\|.$$

(Used in Homotopy Type Theory to define $\exists$ from Σ.)

Continuity in type theory extended with $\| - \|$

Add a universal map $| - | : X \to \|X\|$ into types with at most one element.

The elimination rule is $(X \to P) \to (\|X\| \to P)$

for any type P with at most one element.

(We are quotienting X by the relation that identifies any two points.)

$$\prod_{f\,:\,\mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \; \prod_{\alpha : \mathbb{N}^{\mathbb{N}}} \left\| \sum_{n : \mathbb{N}} \; \prod_{\beta : \mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta \right\| .$$

- In a sheaf topos, this means we can find n locally but not globally.
- In a realizability topos, we can find n intensionally but not extensionally.
- In other toposes this of course acquires other meanings.
- In type theory, it seems difficult to give a direct *meaning-explanation*.

Another well-known example

If you try to say that $f : X \to Y$ is a surjection by saying

$$\prod_{y:Y} \sum_{x:X} fx = y,$$

you are actually saying that f has a section $Y \to X$.

You should instead say

$$\prod_{y:Y} \left\| \sum_{x:X} fx = y \right\|.$$

A similar distinction arises in the definition of the image of a function, and many other definitions and theorems and proofs.

Disclosing secrets

The elimination rule is $(X \to P) \to (\|X\| \to P)$

for any type P with at most one element.

We can disclose a secret $\|X\|$ to P provided we have a map $X \to P$.

Example. If $A(n)$ is decidable then

$$\|\Sigma n : \mathbb{N}.\ A(n)\| \to \Sigma n : \mathbb{N}.\ A(n).$$

Proof sketch. If we have any n with $A(n)$, we can find the minimal n, using the decidability of $A(n)$, but "having a minimal n such that $A(n)$" is a type with at most one element.

More general lemma

From now on everything in the talk is joint work with Chuangjie Xu.

Assume that $A(n)$ has at most one element for every $n : \mathbb{N}$.

If for any given n we have that $A(n)$ implies that $A(m)$ is decidable for all $m < n$, then

$$\|\Sigma n : \mathbb{N}.\ A(n)\| \to \Sigma n : \mathbb{N}.\ A(n).$$

Theorem of MLTT extended with $\| - \|$

$$\Pi f : 2^{\mathbb{N}} \to \mathbb{N}.\ \big\|\Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta\big\| \\ \to \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Proof. Set $A(n) = \big(\Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta\big)$ in the lemma.

Corollary. The topological topos validates the uniform-continuity axiom

$$\Pi f : 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Because the premise of the theorem is validated.

(In the topological topos, the theorem can be seen as getting global existence from local existence by compactness.)

Getting constructive

1. Kleene–Kreisel functionals constructively.

2. Another topological topos for that.

3. If all functions $2^{\mathbb{N}} \to \mathbb{N}$ are continuous, then the Kleene–Kreisel hierarchy agrees with the full-type hierarchy.

4. A model of type theory that constructively validates the uniform-continuity axiom.

5. Implemented in Agda.

Kleene–Kreisel continuous functionals

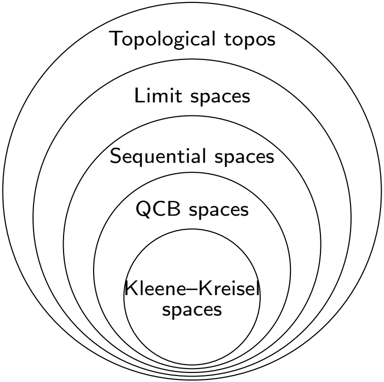
Identified in the 1950's as

- Kleene's countable functionals.
- Kreisel's continuous functionals.

Start from $\mathbb{N}$ and close under exponentiation.

This is automatically closed under finite products, excluding the empty product 1.

Fully abstract model of Gödel's system T.
(By the Kleene-Kreisel density theorem.)



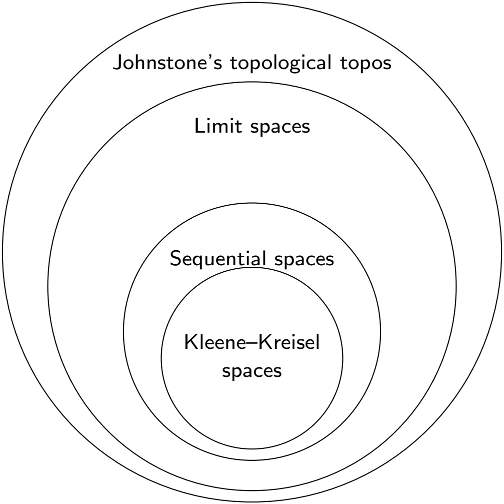
The set-theoretical full type hierarchy is not fully abstract (Kreisel).

Another topological topos

1. Replace $\mathbb{N}_\infty$ by the Cantor space $2^{\mathbb{N}}$.
2. Replace the canonical coverage by the *uniform continuity coverage*.

Amenable to constructive treatment.

Related to Fourman and to van der Hoeven and Moerdijk 1980's.



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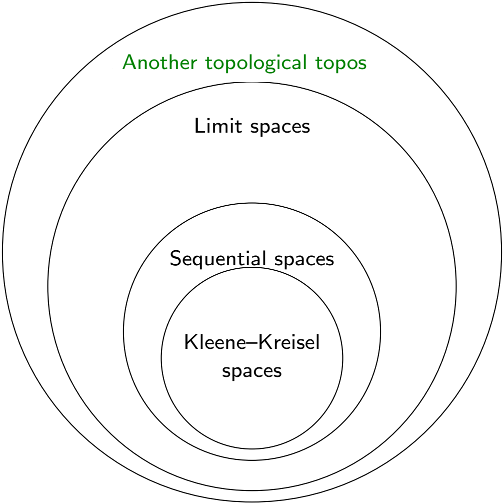

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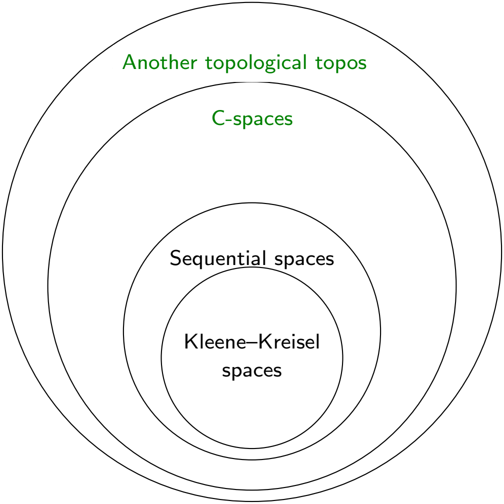
Amenable to constructive treatment.

Related to Fourman and to van der Hoeven and Moerdijk 1980's.



Another topological topos

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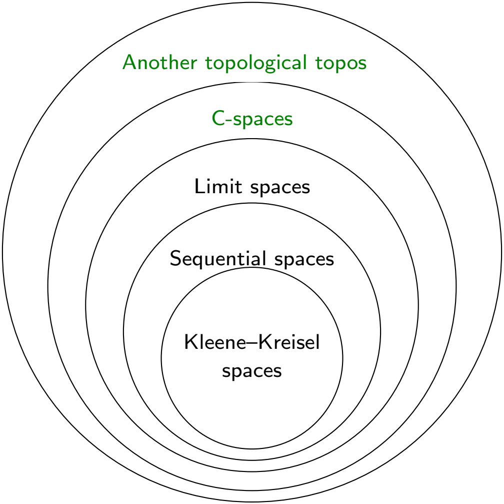
2. Replace the canonical coverage by the *uniform continuity coverage*.

Amenable to constructive treatment.

Related to Fourman and to van der Hoeven and Moerdijk 1980's.



The uniform-continuity coverage

1. Let 2^n denote the set of binary strings of length n.

2. For $s \in 2^n$, let $\mathrm{cons}_s \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ denote the concatenation map

$$\mathrm{cons}_s(\alpha) = s\alpha.$$

3. A function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is uniformly continuous iff

$$\forall m : \mathbb{N}. \exists n : \mathbb{N}. \forall s : 2^n. \exists f' \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}. \exists s' : 2^m. f \circ \mathrm{cons}_s = \mathrm{cons}_{s'} \circ f'.$$

4. This shows that the countable collection $\{(\mathrm{cons}_s)_{s:2^n} \mid n : \mathbb{N}\}$ satisfies the coverage axiom.

5. This coverage is subcanonical.

6. Moreover, crucially: $y\left(2^{\mathbb{N}}\right)$ has the universal property of the exponential $2^{\mathbb{N}}$ in the resulting topos, where of course 2 is $1+1$ and $\mathbb{N}$ is the natural numbers object of the topos.

What we get

1. A constructive treatment of sheaves and C-spaces suitable for development in Martin–Löf type theory.

   Definitions, theorems and proofs implemented in Agda.

   We don't need $\| - \|$.

   We need $\neg\neg$(function extensionality).

2. C-Spaces give a constructive model of dependent types with the uniform continuity axiom.

   At the moment we haven't modelled the universe.

   The amalgamation property for the "naive" version of the Hofmann–Streicher universe holds only up to isomorphism.

   We want to avoid sheafification.

3. If we assume that all functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous, then we can show constructively that the full type hierarchy is equivalent to the Kleene–Kreisel continuous hierarchy.

End