The intrinsic topology of a univalent universe

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A question by Vladimir Voevodsky (17 Oct 2011)

$\boxed{\text{What is the topology of Streicher's universe in the topological topos?}}$

This is not a univalent universe.

$\boxed{\text{In a univalent universe, “miraculously” any two isomorphic types are equal.}}$

Let me begin by putting this question in context.

Topology in constructive mathematics, logic and computation

1. Old story: $\boxed{\text{Secretly, types are } \textit{spaces} \text{ and functions are } \textit{continuous}.}$

   A finite amount of information about the output of a function can only depend on a finite amount of information about the input.

2. New story: $\boxed{\text{Secretly, types are homotopy types, propositional equalities are paths.}}$

   This is just the way things turn out to be when you calculate with them.

Topology in constructive mathematics, logic and computation

1. Old story: Secretly, types are *spaces* and functions are *continuous*.

2. New story: Secretly, types are homotopy types, propositional equalities are paths.

Why are these secrets?

- They cannot be proved or refuted in Bishop or Martin-Löf mathematics.
- They make sense.
- They have models.
- They (should) have (debatable) computation rules.

So better keep them secret!

The universe cannot keep secrets: it is intrinsically indiscrete

Model-independent result.

At least this is so for a univalent universe in Martin-Löf type theory.

This is what we prove in this talk.

Although this talk is not about models,

let me begin by considering well-known models, for the sake of motivation.

An increasing finite sequence of categories

Each embedding in the next category is full.

1. $\boxed{\text{Kleene-Kreisel spaces (1950's).}}$ Continuous model of Gödel's system T.

2. $\boxed{\text{Sequential topological spaces.}}$ A cartesian closed category.

3. $\boxed{\text{Kuratowski limit spaces.}}$ A locally cartesian closed category, a model of MLTT.

4. $\boxed{\text{Johnstone's topological topos (1979).}}$ Model with universe, using Streicher (2004).

   The site is the monoid of continuous endofunctions of the
   one-point compactification of the natural numbers,
   with the canonical topology.

A question by Vladimir Voevodsky (17 Oct 2011)

$\boxed{\text{What is the topology of Streicher's universe in the topological topos?}}$

I will come back to it at the end of the talk.

I formulate and answer a different, related question

1. Work *within* Martin-Löf type theory.

   Informally, but rigorously.

   Like classical mathematicians work within ZFC.

   (But I have written formal proofs, and the computer believes they are correct.)

2. Define intrinsic sequence convergence in the language of Martin-Löf type theory.

Question. What are the intrinsically convergent sequences of types in a universe?

This is a model-independent question.

The answer will be perhaps shocking

In any univalent universe of types:

$$\boxed{\text{Every sequence of types intrinsically converges to any type.}}$$

This means that the universe is intrinsically indiscrete.

(The topology generated by such a convergence relation has only two open sets.)

There can be non-trivial functions only into other *indiscrete spaces*:

1. $U \to U$ and $U \times U \to U$ and $U \times U \to U \times U$ have plenty.
2. $U \to \mathbb{N}$ and $U \to 2$ have only trivial ones, because $\mathbb{N}$ and 2 are *discrete*.

For a not-necessarily-univalent universe

That satisfies the propositional axiom of function extensionality.

(Any two pointwise equal functions are equal, which is a consequence of univalence.)

Theorem. Every sequence of types converges to any type, up to isomorphism.

Dream

$\boxed{\text{A sequence } x_n \text{ converges to a limit } l \text{ if and only if } x_\infty = l.}$

When you wake up in the morning, you realize that it doesn't make sense to write x_∞.

After you have a cup of coffee, you fix this as follows

Extend the type $\mathbb{N}$ to a type $\mathbb{N}_\infty$ so that:

1. The sequence $0, 1, 2, \ldots, n, \ldots$ converges to the limit $\infty \colon \mathbb{N}_\infty$.

2. A sequence $\mathbb{N} \to X$ converges to a limit $x_\infty \colon X$ if and only if it extends to a function

$$\mathbb{N}_\infty \to X$$

   that maps ∞ to x_∞.

$\boxed{\mathbb{N}_\infty \text{ is the generic convergent sequence.}}$

Also called the one-point compactification of $\mathbb{N}$.

The type $\mathbb{N}_\infty$

$\boxed{\text{Taken to be that of decreasing binary sequences.}}$

1. We imagine $1^n 0^\omega \longrightarrow 1^\omega$.
2. We notationally identify

$$\begin{array}{rcl} n & \sim & 1^n 0^\omega, \\ \infty & \sim & 1^\omega. \end{array}$$

3. In Martin-Löf type theory, subtypes are defined using $\sum$:

$$\boxed{\mathbb{N}_\infty = \sum_{\alpha\,:\,2^{\mathbb{N}}} \prod_{n\,:\,\mathbb{N}} (\alpha_n = 0 \to \alpha_{n+1} = 0).}$$

Intrinsic convergence

Definition. A sequence $x\colon \mathbb{N} \to X$ *intrinsically converges* to a limit $x_\infty\colon X$ if and only if it extends to a function

$$\hat{x}\colon \mathbb{N}_\infty \to X$$

that maps ∞ to x_∞.

The topological-topos interpretation of Martin-Löf theory makes this good.

This is just encouragement: We consider this definition independently of any model.

Any function of any two types is sequentially continuous

Without invoking Brouwerian continuity axioms.

Proof. Automatic.

Let $f\colon X \to Y$ be a function.

If $x_n \to x_\infty$ in X, then there is $\mathbb{N}_\infty \to X$ that maps n to x_n and ∞ to x_∞.

Now compose this function $\mathbb{N}_\infty \to X$ with your given function $f\colon X \to Y$.

The resulting function $\mathbb{N}_\infty \to Y$ maps n to $f(x_n)$ and ∞ to $f(x_\infty)$.

This means that $f(x_n) \to f(x_\infty)$.

Q.E.D.

Convergence up to isomorphism in a universe U

Definition. We say that a sequence of types

$$X : \mathbb{N} \to U$$

intrinsically converges to a limit $X_\infty : U$ if and only if it extends to a function

$$\hat{X} : \mathbb{N}_\infty \to U$$

with

$$\hat{X}_n \cong X_n, \qquad \hat{X}_\infty \cong X_\infty.$$

For a univalent universe, this definition is equivalent to the previous one.

Because type isomorphism is equivalent to type equality in such a universe.

Theorem. The universe is intrinsically indiscrete

Assuming the axiom of function extensionality:

Every sequence of types converges to any type, up to isomorphism.

Warming-up exercise

The constant sequence of types $(0)_n$ converges to 1.

We write $(0)_n \longrightarrow 1$.

Proof. Define $X : \mathbb{N}_\infty \to U$ by

$$X_u = \forall k : \mathbb{N}(u \neq k) \quad = \prod_{k : \mathbb{N}} (\mathrm{Id}\, \mathbb{N}_\infty\, u\, (\mathrm{incl}\, k) \to 0).$$

Then, for any $n : \mathbb{N}$,

$$X_n \cong 0,$$

and

$$X_\infty \cong 1.$$

Extensionality is used to have these isomorphisms.

Lemma. Every sequence of types converges to the type $\mathbf{1}$

Given $X : \mathbb{N} \to U$, define $\hat{X} : \mathbb{N}_\infty \to U$ by

$$\boxed{\hat{X}_u = \prod_{k \,:\, \mathbb{N}} (u = k \to X_k).}$$

Then

$$\hat{X}_n = \prod_{k \,:\, \mathbb{N}} (n = k \to X_k) \cong X_n,$$

and

$$\hat{X}_\infty = \prod_{k \,:\, \mathbb{N}} (\infty = k \to X_k) \cong \prod_{k \,:\, \mathbb{N}} (\mathbf{0} \to X_k) \cong \prod_{k \,:\, \mathbb{N}} \mathbf{1} \cong \mathbf{1}.$$

This argument requires subtle identity-type inductions.

Theorem. Every sequence of types converges to any type

Let $X : \mathbb{N} \to U$ and $Y : U$ be given.

(i) $\boxed{X_n \longrightarrow \mathbf{1}}$ Previous lemma.

(ii) $\boxed{(\mathbf{0})_n \longrightarrow \mathbf{1}}$ Special case of (i).

(iii) $\boxed{(\mathbf{0})_n \longrightarrow Y}$ Multiply (ii) by Y, and use $\mathbf{0} \times Y \cong \mathbf{0}$ and $\mathbf{1} \times Y \cong Y$.

(iv) $\boxed{(\mathbf{1})_n \longrightarrow \mathbf{0}}$ Compose (ii) with $(- \to \mathbf{0})$ and use $(\mathbf{0} \to \mathbf{0}) \cong \mathbf{1}$ and $(\mathbf{1} \to \mathbf{0}) \cong \mathbf{0}$.

(v) $\boxed{X_n \longrightarrow \mathbf{0}}$ Multiply (i) and (iv), and use $X \times \mathbf{1} \cong X$ and $\mathbf{1} \times \mathbf{0} \cong \mathbf{0}$.

(vi) $\boxed{X_n \longrightarrow Y}$ Add (iii) and (v), and use $\mathbf{0} + X_n \cong X_n$ and $Y + \mathbf{0} \cong Y$.

Corollary. Rice's Theorem for the Universe

It is a taboo to say that the universe has a non-trivial, extensional, decidable property.

For any extensional $P\colon U \to 2$ and $X, Y : U$, if $P(X) \neq P(Y)$ then WLPO.

We say that $P\colon U \to 2$ is extensional if $X \cong Y \implies P(X) = P(Y)$.

(For a univalent universe, all decidable properties are extensional.)

WLPO is the statement that every binary sequence is constantly 1 or it isn't.

$$\boxed{P\colon U \to 2 \textbf{ extensional},\ X, Y : U, \quad P(X) \neq P(Y) \implies \textbf{WLPO}}$$

Assume w.l.o.g. that $P(X) = 0$ and $P(Y) = 1$.

By the Universe Indiscreteness Theorem, there is $Q\colon \mathbb{N}_\infty \to U$ with

$$\forall n\colon \mathbb{N}(Q(n) \cong X), \qquad Q(\infty) \cong Y.$$

Let $p\colon \mathbb{N}_\infty \to 2$ be $P \circ Q$. By the extensionality of P, we have that

$$\forall n\colon \mathbb{N}(p(n) = 0), \qquad p(\infty) = 1.$$

Hence for any given $x\colon \mathbb{N}_\infty$ we can decide whether $x = \infty$ by checking the decidable condition

$$p(x) = 1.$$

This amounts to WLPO.

It is decidable whether every binary sequence is constanly 1.

Sharper version of Rice's Theorem for the Universe

Found by Alex Simpson after I gave a similar talk two weeks ago at MFPS'2012.

T.F.A.E.

1. There is a non-trivial, extensional map $U \to 2$.
2. $\forall P : U(\neg P \vee \neg\neg P)$.

Alex's proof is nice and short, and doesn't rely on our proof of the Universe Indiscreteness Theorem.

But it does rely, like ours, on the axiom of function extensionality for small types.

But let's not neglect Voevodsky's original question

Streicher's universe U in the topological topos is not univalent.

So our argument doesn't apply

Conjecture

Encouraged by the above, I posed the following to Thomas Streicher:

$$\boxed{\text{In the topological topos, all maps of } U \text{ into the Sierpinski space } \mathbb{S} \text{ are constant.}}$$

The Sierpinski space $\mathbb{S}$ is the open-subspace classifier.

It has an isolated point $\top$ and a limit point $\perp$.

Hence this conjecture would imply that that U has only two open subspaces.

Theorem (Streicher). The conjecture is indeed true

$\boxed{\text{Streicher's universe in the topological topos has only two open subspaces.}}$

Streicher's universe is topologically indiscrete.

Challenge: reconcile the old and new topological views

1. (Old) Types are spaces, functions are continuous.
2. (New) Types are homotopy types, propositional equalities are paths.

Types should be $\boxed{\text{topological ?-groupoids.}}$

1. As a space, $2^{\mathbb{N}}$ is rather rich, but as a groupoid it is trivial.
2. As a space, U is trivial, but as a groupoid it is rather rich.
3. Then consider something like $2^{\mathbb{N}} \times U$.

Task. Come up with a model (e.g. simplicial objects in the topological topos) that simultaneously accounts for the old and new topological views.

More importantly, understand this in a model-independent way.
With and without suitable additional axioms.